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CONSTRUCTION OF COHOMOLOGY OF DISCRETE GROUPS

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ABSTRACT. A correspondence between Hermitian modular forms and vector valued harmonic forms in locally symmetric spaces associated to $U(p,q)$ is constructed and also shown in general to be nonzero. The construction utilizes Rallis-Schiffmann type theta functions and simplified arguments to circumvent differential geometric calculations used previously in related problems.

Introduction. In our previous papers [**T.W.2–4**, **W.1**, **2**] and that of Kudla-Millson [**K.M.1**, **2**] various cases are proved about correspondences of harmonic forms on locally symmetric spaces dual to geodesic cycles on the one hand, and Siegel, Hermitian, or quaternionic modular forms on the other hand. The existence of such correspondences is based on R. Howe's theory of decomposition of the oscillator representation on reductive dual pairs [**Ho.1**, **2**]. However, while the general theory has a simple representation theoretic description, cf. [**H.-P.S.**, §2], this description is pretty much lost in the previous works on correspondences of cycles and modular forms. This is mainly due to technicalities in constructing harmonic forms and then passing from harmonic forms to cycles.

The purpose of this paper is twofold. First of all in terms of results we generalize [**T.W.4**] to $U(p,q)$, and in fact we generalize to harmonic forms with coefficients in locally constant vector bundles. We also prove these harmonic forms can be nonzero. Secondly, in terms of methodology, and this is perhaps an equally significant point, we show that the correspondence of cycles and modular forms has a straightforward representation theoretic description. This description consists of the following ingredients.

1. Construction of some functions on the matrix space $M_{p+q,r}(\mathbf{C})$ which are in the discrete spectrum of the reductive dual pair $U(p,q) \times U(r,r)$ and are $(U(p) \times U(q)) \times (U(r) \times U(r))$ finite. These functions are of highest weight for $U(r,r)$ and are generalizations of the Rallis-Schiffmann functions [**R.S.1**, **2**, **L.V.**]. These functions determine some pieces of the representation correspondence. If convergence holds, the theta distribution applied to such functions then gives the theta kernels used for "global" correspondence. It is an easy argument [**B.W.**, VIII] to show that the theta kernels are in general nonzero.

2. Construction of an intertwining map $\omega_X^\lambda(Z)$, $X \in M_{p+q,r}(\mathbf{C})$, $Z \in \mathcal{D} = U(p,q)/(U(p) \times U(q))$, from the representation space of $U(p,q)$ in 1 above to a space of harmonic forms on $\mathcal{D}$ with coefficients in a locally constant bundle E_λ.

3. For an arithmetic discrete subgroup $\Gamma \subset G = U(p,q)$ determination of the cohomology class of $\hat{\omega}_X^\lambda$ on $\Gamma \setminus \mathcal{D}$ where $\hat{\omega}_X^\lambda = \sum_{\gamma \in \Gamma_X \setminus \Gamma} \gamma^* \hat{\omega}_X^\lambda$, $\Gamma_X = \Gamma \cap G_X$, and

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$G_X \subset G$ fixes X. Thus for any E_λ^* valued harmonic form φ on $\Gamma \setminus \mathcal{D}$ we prove

$$\int_{\Gamma\setminus\mathcal{D}} \hat{\omega}_X^\lambda \wedge \varphi = \int_{C_X} \sigma_X^\lambda(\varphi|_{C_X})$$

where σ_X^λ is a section of E_λ over a complex geodesic cycle $C_X \subset \Gamma \setminus \mathcal{D}$. By Serre duality the cohomology class of $\hat{\omega}_X^\lambda$ is then determined. Previously in the case $E_\lambda = \mathbf{C}$ this equality was proved using Chern's trangression formulas and relating ω_X to geometrically constructed dual forms. All these required extensive calculations. In this paper, we show that the equality is in fact a consequence of an invariant characterization of ω_X^λ (Theorem 4.8) and a simple pairing argument (Lemma 5.6).

The interpretation of liftings of higher weight modular forms as cohomology classes represented (in the language of currents) by sections of bundles supported on cycles have also appeared in Hilbert modular surfaces [**T**]. In that case, another interpretation [**G**] is that the higher weight forms correspond to cycles in certain Kuga fiber spaces. It is likely that similar interpretations are possible for the harmonic forms constructed here.

Due to convergence problems our nonvanishing results do not include the case $E_\lambda = \mathbf{C}$. If convergence were to hold we note that

$$0 \neq \hat{\omega}_X^\lambda(Z) = \pi_Z(\mathcal{H}(C_X))$$

where $\mathcal{H}(C_X)$ is the harmonic form Poincaré dual to the algebraic cycle C_X and π_Z is the pointwise projection to the K_Z (isotropy subgroup at Z) irreducible subspace of weight

$$(\underbrace{q\cdots q}_{r}, 0\cdots 0, \underbrace{-q\cdots -q}_{r})(\underbrace{0\cdots 0}_{q}).$$

It would then be an interesting question to decide if $\hat{\omega}_X^\lambda(Z)$ (or other similar harmonic forms with a more complicated K type) is dual to an algebraic cycle.

In [**T.W.5**] we showed a much wider range of nonvanishing cohomology on $\Gamma \setminus \mathcal{D}$. In the dual pair correspondence and on the level of representations these are essentially all the ones that correspond to holomorphic discrete series of $U(r,r)$. It is interesting to see if geometric representatives for these cohomology can also be found.

As representations of $SU(p,q)$ the spaces of harmonic forms considered here and their natural generalizations are Flensted-Jensen representations [**F.1**] for a semisimple symmetric space G/H. ($H = S(U(r) \times U(p-r,q))$ for this paper.) In particular, the uniqueness in Theorem 4.8 is closely related to the fact that these representations should have multiplicity 1 in $L^2(G/H)$ [**F.2**]. We are, however, unable to make use of the integral formula of Flensted-Jensen functions for present purposes since it is given in the dual G^0/H^0 while we need the functions on the matrix space $M_{p+q,r}(\mathbf{C})$.

It may be relevant to mention that in [**K.M.2**] by quite different techniques certain representatives φ of continuous cohomology classes are constructed which in the present context of unitary groups give $[\varphi] \in H_{ct}^*(U(p,q), \mathcal{S}(M_{p+q,r}(\mathbf{C}))_\alpha)$, where $\mathcal{S}$ indicates the Schwartz space and the subscript α means twisting by a character. A natural question (a question to that effect is posed in [**K.M.2**]) is to determine the representation spaces of $U(p,q)$ and $U(r,r)$ spanned by the translates of φ. By comparing the cohomology degrees it is easy to see that the space discussed in 1 above which is spanned by $\omega_X(Z)e^{-2\pi\,\mathrm{tr}(X,X)}$ (cf. §1.1 for notation) should give a distinguished subspace in that of φ.

The methods used here also work for the reductive dual pairs $(O(p,q), Sp(r,\mathbf{R}))$, $(Sp(p,q), O^*(2r))$. Our results also extend to compact quotients, the readers may consult [**T.W.2**, **5**, **W.1**] for the details to formulate such results. We should emphasize that in these papers as well as in [**K.M.1**, **2**] the theta kernel is always gotten by summing a Schwartz function, but the Schwartz functions are not in the discrete spectrum. The Rallis-Schiffmann functions are not Schwartz functions, thus some care is needed in summing, but it has considerable advantages in representation questions since it is in the discrete spectrum. We are indebted to [**L.V**] for a thorough exposition of the approach of Rallis-Schiffmann.

1. Some functions in the discrete spectrum of $U(p,q)\times U(r,r)$.

1.1. We define certain functions in the discrete spectrum of the dual pair $U(p,q)\times U(r,r)$ and compute the eigenvalues of Casimir operators acting on them. The functions are generalizations of the Rallis-Schiffmann functions studied in [**R.S.1**, **2**] for the pair $O(p,q)\times SL(2,\mathbf{R})$.

Let $G=U(p,q)$ $(p+q=n)$ and let $\mathfrak{g}$ be its Lie algebra. G leaves invariant the Hermitian form on $\mathbf{C}^n$ given by

$$(x,y)={}^t\bar{x}\begin{pmatrix}E_p & \\ & -E_q\end{pmatrix}y.$$

Let $V=M_{nr}(\mathbf{C})$ denote the space of $n\times r$ complex matrices. Let $X=(X_{ij})\in V$ be the usual row and column coordinates and let X_j denote the jth column of X. We let

$$\frac{\partial}{\partial X_j}=\begin{pmatrix}\vdots\\ \dfrac{\partial}{\partial X_{ij}}\\ \vdots\end{pmatrix},\qquad 1\le i\le n,$$

$$L_{ij}=\sum_{k=1}^n X_{ki}\frac{\partial}{\partial X_{kj}},\qquad \overline{L}_{ij}=\sum_{k=1}^n \overline{X}_{ki}\frac{\partial}{\partial \overline{X}_{kj}},$$

and

$$\left(\frac{\partial}{\partial X_i},\frac{\partial}{\partial X_j}\right)=\sum_{k=1}^p\frac{\partial^2}{\partial\overline{X}_{ki}\partial X_{kj}}-\sum_{k=p+1}^n\frac{\partial^2}{\partial\overline{X}_{ki}\partial X_{kj}}.$$

Let l be the left action of G on the smooth function of V, i.e.

$$l(g)f(x)=f(g^{-1}x),\qquad g\in G,\ X\in V.$$

Then the induced action of the Casimir $C(p,q)$ of G is given by [**T.W.5**, Lemma 1.12].

$$l(C(p,q))=\sum_{1\le i,j\le r}(L_{ij}L_{ji}+\overline{L}_{ij}\overline{L}_{ji})+(n-r)\sum_i(L_{ii}+\overline{L}_{ii})\\ -2\sum_{1\le i,j\le r}(X_i,X_j)\left(\frac{\partial}{\partial X_i},\frac{\partial}{\partial X_j}\right).\tag{1}$$

1.2. Let $GL(p,\mathbf{C})\times GL(r,\mathbf{C})\times GL(r,\mathbf{C})$ act on functions on $M_{pr}(\mathbf{C})\times M_{pr}(\mathbf{C})$ by

$$(g,A,B)f(x,y)=f(g^{-1}xA,{}^tgy^tB^{-1}). \tag{1}$$

Let

$$f_{a,b}(x,y)=\det\begin{pmatrix} x_{11} & \cdots & x_{1r}\\ \vdots & & \vdots\\ x_{r1} & \cdots & x_{rr}\end{pmatrix}^a \det\begin{pmatrix} y_{p-r+11} & \cdots & y_{p-r+1,r}\\ \vdots & & \vdots\\ y_{p1} & \cdots & y_{pr}\end{pmatrix}^b; \tag{2}$$

then $f_{a,b}$ is a pluriharmonic polynomial [**K.V.**, (6.1)]. On the set of pluriharmonic polynomials the representation (1) is the complexification of the representation of $U(p)\times U(r)\times U(r)$ acting by

$$(g,A,B)f(X,\overline{X})=f(g^{-1}XA,\overline{g^{-1}XB}). \tag{3}$$

We parametrize the irreducible representation of $GL(p,\mathbf{C})$ by its highest weight w.r.t lower triangular Borel subgroups and that of $GL(r,\mathbf{C})\times GL(r,\mathbf{C})$ w.r.t. product of upper triangular Borel subgroups. Then $f_{a,b}(x,y)$ is a highest weight vector with weight

$$(\overbrace{b,\ldots,b}^{r},0,\ldots,0,\overbrace{-a,\ldots,-a}^{r})\quad \text{for } GL(p,\mathbf{C}) \tag{4}$$

and

$$(a,\ldots,a)\otimes(-b,\ldots,-b)\quad \text{for } GL(r,\mathbf{C})\times GL(r,\mathbf{C}). \tag{5}$$

1.3. For $X\in M_{nr}(\mathbf{C})$, let $X_+=(X_{ij})$, $1\le i\le p$. Thus $X_+\in M_{pr}(\mathbf{C})$. Also let X_{+j} be the jth column of X_+. (X_+,X_+) then denotes the $r\times r$ matrix whose (i,j)th entry is ${}^t\overline{X}_{+i}X_{+j}$. We shall consider the action of $l(C(p,q))$ on the function

$$f_{a,b,s}(X)=\det(X_+,X_+)^s f_{a,b}(X_+,\overline{X}_+). \tag{1}$$

First of all since $f_{a,b,s}$ is invariant under $SU(r)$,

$$L_{ij}f_{a,b,s}=0 \quad\text{and}\quad \overline{L}_{ij}f_{a,b,s}=0,\qquad i\ne j.$$

Next by counting degrees

$$L_{ii}f_{a,b,s}=r(a+s),\quad \overline{L}_{ii}f_{a,b,s}=r(b+s). \tag{2}$$

It remains to compute the action of $(\partial/\partial X_i,\partial/\partial X_j)$. Let $\mathrm{Adj}(X_+,X_+)_{ij}$ be the (i,j)th entry in the adjoint matrix of (X_+,X_+).

LEMMA.

$$\left(\frac{\partial}{\partial X_i},\frac{\partial}{\partial X_j}\right)\det(X_+,X_+)^s=s(p-r+s)\det(X_+,X_+)^{s-1}\,\mathrm{Adj}(X_+,X_+)_{ij}.$$

PROOF. A direct calculation gives

$$\begin{aligned}\left(\frac{\partial}{\partial X_i},\frac{\partial}{\partial X_j}\right)\det(X_+,X_+)^s&=s(s-1)\det(X_+,X_+)^{s-2}\\ &\quad\times\sum_{k,l}(X_{+k},X_{+l})\,\mathrm{Adj}(X_+,X_+)_{ik}\,\mathrm{Adj}(X_+,X_+)_{jl}\\ &\quad+s(p-r+1)\det(X_+,X_+)^{s-1}\,\mathrm{Adj}(X_+,X_+)_{ij}.\end{aligned}$$

Since $(\partial/\partial X_i, \partial/\partial X_j)$ annihilates the pluriharmonic function $f_{a,b}(X,\overline{X})$ we have by straightforward calculation using the lemma:

$$(3)\qquad \left(\frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j}\right) f_{a,b,s}(X) = s(a+b+p-r+s) f_{a,b}(X_+, \overline{X}_+) \times \operatorname{Adj}(X_+, X_+)_{ij} \det(X_+, X_+)^{s-1}.$$

In particular $f_{a,b,s}$ is pluriharmonic if $s = -(a+b+p-r)$.

PROPOSITION. *Let* $s = -(a+b+p-r)$. *Then*

$$l(C(p,q)) f_{a,b,s} = r\{(p-r+a)(a-q) + (p-r+b)(b-q)\} f_{a,b,s}.$$

PROOF. $f_{a,b,s}$ has degree $-(p-r+a)$ in X_+ and degree $-(p-r+b)$ in $\overline{X}_+$. The result is then a consequence of 1.1(1), 1.3(3) and the remark preceding the proposition.

COROLLARY. *Let $h(X_+)$ be a pluriharmonic polynomial in the space with highest weight* 1.2(4) *and let*

$$s = -(a+b+p-r).$$

Then $l(C(p,q))$ acting on $h(X_+)\det(X_+, X_+)^s$ has the same eigenvalue as in the proposition.

PROOF. $h(X_+)$ is also annihilated by $(\partial/\partial X_i, \partial/\partial X_j)$ and has the same homogeneous degrees as $f_{a,b}$. The same computations can be applied.

1.4. For $X \in V$, let (X,X) be the $r \times r$ matrix whose (i,j)th entry is (X_i, X_j). We consider the function

$$(1)\qquad \varphi_{a,b,t}(X) = \det(X,X)^t f_{a,b,s}(X) \quad \text{where } s = -(a+b+p-r).$$

Since $\det(X,X)$ is a G-invariant function, $\varphi_{a,b,t}$ remains as an eigenfunction for $l(C(p,q))$ with the same eigenvalue as in Proposition 1.3. We are interested in finding values of $t \neq 0$, such that $\varphi_{a,b,t}$ is pluriharmonic. Using the fact that $f_{a,b,s}$ is pluriharmonic, a calculation identical to 1.3(3) and Lemma 1.3 shows that

$$\left(\frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j}\right) \varphi_{a,b,t}(X) = t(-p+q+r+t-a-b) \times \det(X,X)^{t-1} \operatorname{Adj}(X,X)_{ij} f_{a,b,s}(X).$$

Thus $\varphi_{a,b,t}(X)$ is pluriharmonic if $t = p-q-r+a+b$. We introduce the following definition:

$$(2)\qquad \varphi_{a,b}(X) = \begin{cases} \varphi_{a,b,t}(X), & t = p-q-r+a+b \text{ if } (X,X) > 0, \\ 0 & \text{if } (X,X) \not> 0. \end{cases}$$

1.5. The generalized Rallis-Schiffmann function is $\varphi_{a,b}(X)e^{-2\pi \operatorname{tr}(X,X)}$. It will be shown in §2 that this function is in $L^2(V)$ for a, b sufficiently large.

As in [**T.W.5**, §3] let

$$J_r = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},$$

and

$$U(J_r) = \{g \in GL(2r, \mathbf{C}) | g^* J g = J\}.$$

For

$$\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} I_r & \sqrt{-1}I_r \\ \sqrt{-1}I_r & I_r \end{pmatrix},$$

a maximal compact subgroup of $U(J_r)$ is $\{\kappa(x,y) = \alpha(x,y)\alpha^{-1} | (x,y) \in U(r) \times U(r)\}$. Let $r_0(g)$, $g \in U(J_r)$, be the modified Weil (oscillator) representation acting on $L^2(V)$ [**T.W.5**, §9.2]. We shall also denote by $r_0(X)$ the induced operator for X in the Lie algebra of $U(J_r)$.

LEMMA. *Let $F(X, \overline{X})$ be a pluriharmonic function. Then*

$$r_0(\kappa(x,y))F(X,\overline{X})e^{-2\pi \operatorname{tr}(X,X)} = F(Xx, \overline{X}^t y^{-1}) \det(y)^{-n} e^{-2\pi \operatorname{tr}(X,X)}.$$

PROOF. This is an analogue of [**T.W.5**, Proposition 9.10] and proved in the same manner. Note the only difference is that we have here $e^{-2\pi \operatorname{tr}(X,X)}$ instead of the $e^{-2\pi \operatorname{tr}(X,X)_0}$ in [**T.W.5**, 9.5(1)]. Then [**T.W.5**, Lemma 9.9] should be replaced by: for $a^* = -a \in M_{r,r}(\mathbf{C})$

$$r_0(\kappa(a,0))F(X,\overline{X})e^{-2\pi \operatorname{tr}(X,X)} = F(Xa, \overline{X})e^{-2\pi \operatorname{tr}(X,X)}$$

and

$$r_0(\kappa(0,a))F(X,\overline{X})e^{-2\pi \operatorname{tr}(X,X)} = \{-n \operatorname{tr}(a) F(X,\overline{X}) + F(X, \overline{X}a^*)\} e^{-2\pi \operatorname{tr}(X,X)}.$$

From these the lemma follows.

COROLLARY. *Let $h(X_+)$ be as in Corollary* 1.3 *and*

$$\varphi(X) = h(X_+) \det(X_+, X_+)^{-(a+b+p-r)} \det(X,X)^{a+b+p-r-q}.$$

Then

$$r_0(\kappa(x,y))\varphi(X)e^{-2\pi \operatorname{tr}(X,X)} = \det(x)^{a-q} \det(y)^{-n-b+q} \varphi(X) e^{-2\pi \operatorname{tr}(X,X)}.$$

2. Square integrability and summability properties.

2.1. We examine the square integrability of the function $\varphi_{a,b}(X)e^{-2\pi \operatorname{tr}(X,X)}$.

PROPOSITION. *If $a + b \geq 2q$, then $\varphi_{a,b}(X)e^{-2\pi \operatorname{tr}(X,X)} \in L^2(V)$.*

PROOF. Let $X = \binom{X_+}{X_-} \in V$, $X_+ \in M_{pr}(\mathbf{C})$, $X_- \in M_{qr}(\mathbf{C})$ and let $\{dX_+\}$ ($\{dX_-\}$) be the Euclidean volume elements for $M_{pr}(\mathbf{C})$ ($M_{qr}(\mathbf{C})$). Since by [**T.W. 5**, §4]

$$|f_{a,b}(X,X)|^2 \leq \det(X_+, X_+)^{a+b}$$

the square integrability of $\varphi_{a,b}(X)e^{-2\pi \operatorname{tr}(X,X)}$ will follow from the convergence of the integral

$$(1) \qquad \int_{\substack{X \in V \\ (X,X)>0}} \det(X_+,X_+)^{\sigma} \det(X,X)^{\tau} e^{-4\pi \operatorname{tr}(X,X)} \{dX_+\}\{dX_-\}$$

where $\sigma = -2(p-r) - (a+b)$ and $\tau = 2(p-q-r+a+b)$.

Let $\varsigma = (X_+, X_+)$ and (X_-, X_-) be the $r \times r$ matrix ${}^t\overline{X}_- X_-$. By Lemma A.1 of the appendix (1) equals

$$(2) \qquad \int_{\substack{\varsigma - (X_-,X_-)>0 \\ X_- \in M_{qr}(\mathbf{C})}} \det(\varsigma)^{\sigma+p-r} \det(\varsigma - (X_-,X_-))^{\tau} e^{-4\pi \operatorname{tr}(\varsigma - (X_-,X_-))} \, d\varsigma \, \{dX_-\}, \qquad X_- \in M_{qr}(\mathbf{C}).$$

We now divide into two cases.

Case 1. $q \geq r$. Let $\xi = (X_-, X_-)$. Then using Lemma A.1 one more time we are reduced to

$$(3)\qquad \int_{\varsigma > \xi \geq 0} \det(\varsigma)^{\sigma+p-r} \det(\varsigma - \xi)^{\tau} e^{-4\pi \operatorname{tr}(\varsigma-\xi)} \det(\xi)^{q-r}\, d\varsigma\, d\xi$$
$$= \int_{\substack{\xi\geq 0\\ \eta>0}} \det(\eta+\xi)^{\sigma+p-r} \det(\eta)^{\tau} e^{-4\pi \operatorname{tr}\eta} \det(\xi)^{q-r}\, d\eta\, d\xi.$$

Consider

$$(4)\qquad \int_{\xi\geq 0} \det(\xi)^{q-r} \det(\eta+\xi)^{\sigma+p-r}\, d\xi.$$

Since $q \geq r$ the integrand is continuous at $\det \xi = 0$. And since $p \geq q$, $a+b \geq 2q$, we have

$$\sigma + p + q - 2r = -2(p-r) - (a+b) + p + q - 2r \leq -(a+b) \leq -2r.$$

The integral in (4) converges by [**H**, p. 38]. Make the change of variable $\xi = \eta\mathcal{Y}$; then (4) is given by $c\det(\eta)^{q-r-(p-r)-(a+b)}$. The convergence of (3) now reduces to that of

$$\int_{\eta>0} \det(\eta)^{p-2r+a+b-q} e^{-4\pi \operatorname{tr}\eta}\, d\eta.$$

Since $p \geq 2r$ and $a+b > q$ the proof is complete.

Case 2. $q < r$. As before $(X_-, X_-) = {}^t\overline{X}_- X_-$, but let $\xi = X_- {}^t\overline{X}_-$ (a $q \times q$ matrix) and $\eta = \varsigma - (X_-, X_-)$. Then (2) reduces to

$$(5)\qquad \int_{\substack{\xi\geq 0\\ \eta>0}} \det(\varsigma)^{\sigma+p-r} \det(\eta)^{\tau} e^{-4\pi \operatorname{tr}\eta} \det(\xi)^{r-q}\, d\eta\, d\xi.$$

Since $\eta > 0$ and $(X_-, X_-) \geq 0$ we may simultaneously diagonalize both matrices. And since (X_-, X_-) has rank $\leq q$, we clearly have $\det(\varsigma) \geq \det(\eta)$ and $\det(\varsigma) \geq P(\eta)\det(\xi)$, where $P(\eta)$ is a monomial in the eigenvalues of η of degree $r-q$. We split the domain $\xi \geq 0$ into $I > \xi \geq 0$ and $\xi \geq I$; then

$$\int_{\xi\geq 0} \det(\varsigma)^{\sigma+p-r} \det(\xi)^{r-q}\, d\xi$$
$$\leq C\det(\eta)^{\sigma+p-r} + \int_{\xi\geq I} \det(\varsigma)^{\sigma+p-r} \det(\xi)^{r-q}\, d\xi$$
$$\leq C\det(\eta)^{\sigma+p-r} + P(\eta)^{\sigma+p-r} \int_{\xi\geq I} \det(\xi)^{\sigma+p-q}\, d\xi.$$

The convergence of $\int_{\xi\geq I} \det(\xi)^{\sigma+p-q}\, d\xi$ is as in Case 1. It suffices then to consider $\int_{\eta>0} \det(\eta)^{\tau+\sigma+p-r} e^{-4\pi \operatorname{tr}\eta}\, d\eta$. But $\tau + \sigma + p - r = a + b - 2q + p - r$ where $p \geq 2r$ and $a + b \geq 2q$, the convergence of the integral follows.

REMARK. It can be shown that under the hypothesis $a+b > 2q$ the function $\varphi_{a,b}(X)e^{-\operatorname{tr}(X,X)}$ is continuous (cf. [**L.V.**, p. 228]).

2.2. Let k be a purely imaginary quadratic field over $\mathbf{Q}$. Let W_k be a $p+q$ dimensional vector space over k (,); $W_k \times W_k \to K$ a nondegenerate Hermitian form conjugate linear in the first variable. Let $W = W_k \otimes_{\mathbf{Q}} \mathbf{R}$ and (,) is extended to a Hermitian form on W. We assume its signature to be (p, q). Let $\mathcal{O}$ be the ring of integers of k and L_0 an $\mathcal{O}$ lattice of W_k, such that $(L_0, L_0) \subset \mathcal{O}$. Then L_0 is contained in its dual lattice

$$L_0^* = \{w \in W_k \,|\, \operatorname{tr}_{k/\mathbf{Q}}(w, L_0) \subseteq \mathbf{Z}\}.$$

Now let $V = W^r$, L be a multiple of L_0^r, and

$$L^* = \{v \in V \,|\, \operatorname{tr}_{\mathbf{C}/\mathbf{R}} \operatorname{tr}(v, L) \subseteq \mathbf{Z}\}.$$

It follows by our hypothesis that $L \subset L^*$. We may choose a basis $\{e_1, \dots, e_n\}$ in W_k to diagonalize $(\, , \,)$ so that it is positive definite on $W_+ = \langle e_1, \dots, e_p\rangle \subset W$, and negative definite on $W_- = \langle e_{p+1}, \dots, e_n\rangle \subset W$. Using this basis we identify V with $M_{nr}(\mathbf{C})$. $V_+ = W_+^r$ with $M_{pr}(\mathbf{C})$ and $V_- = W_-^r$ with $M_{qr}(\mathbf{C})$. We also have $L = L_+ \oplus L_-$, $L_+ = L \cap V_+$ and $L_- = L \cap V_-$.

2.3. Some lemmas are proved here and in the next section to be used later for discussions on summability.

LEMMA. (i) *Suppose A, B are $r \times r$ Hermitian matrices such that $A > 0$ and $B \geq 0$. Then*

$$\frac{\det A}{\operatorname{tr} A} \operatorname{tr}(A + B) \leq \det(A + B).$$

(ii) *Let A be as in* (i). *Then for $\sigma > qr$*

$$\sum_{X_- \in L_-} \operatorname{tr}(A + (X_-, X_-))^{-\sigma} \leq C \operatorname{tr}(A)^{-\sigma + qr},$$

where C is a constant independent of A.

PROOF. (i) This is equivalent to

$$\operatorname{tr}(B) \leq \operatorname{tr}(A)\{\det(I + A^{-1}B) - I\}.$$

We have

$$\det(I + A^{-1}B) = \sum_{j \geq 0} \operatorname{tr} \bigwedge^j (A^{-1}B),$$

and since $A^{-1}B \geq 0$, $\operatorname{tr} \bigwedge^j (A^{-1}B) \geq 0$. It suffices then to show that

$$\operatorname{tr}(B) \leq \operatorname{tr}(A) \operatorname{tr}(A^{-1}B).$$

But we can find $R \in GL(r, \mathbf{C})$ such that $A = {}^t\overline{R}R$, $B = {}^t\overline{R}DR$, where $D \geq 0$ is a diagonal matrix. Then $\operatorname{tr}(A^{-1}B) = \operatorname{tr}(D)$ and the above inequality follows readily.

(ii) The series can be compared with the integral

$$\int_{M_{qr}(\mathbf{C})} \left(\operatorname{tr} A + \sum_{i,j} |X_{ij}|^2\right)^{-\sigma} \{dX_{ij}\}$$

for which the desired inequality follows by a change of variable.

2.4. LEMMA. (i) *Let A be as in the preceding lemma. Then*

$$\sum_{\substack{X_+ \in L_+ \\ (X_+, X_+) = A}} \det(X_+, X_+)^{-\sigma} < C \det(A)^{-\sigma + p}.$$

(ii) *Let A be as above, and $Y_+ \in V_+$. Then*

$$\sum_{\substack{X_+ \in L_+ \\ (X_+ + Y_+, X_+ + Y_+) = A}} \det(X_+ + Y_+, X_+ + Y_+)^{-\sigma} < C \det(A)^{-\sigma+p+r}.$$

PROOF. (i) The Hermitian form (,) restricted to L_+ is positive definite. It is a standard fact that the numbers $\mathrm{Card}\{X_+ \in L_+, (X_+, X_+) = A\}$ are the Fourier coefficients of a Hermitian modular form of genus r and weight p. The inequality $\mathrm{Card}\{X_+ \in L_+, (X_+, X_+) = A\} \leq C \det(A)^p$ then follows from usual estimates (cf. [**Ma**]).

(ii) We have

$$(X_+, X_+) - (Y_+, Y_+) \leq (X_+ + Y_+, X_+ + Y_+) \leq (X_+, X_+) + (Y_+, Y_+).$$

It follows that

$$\begin{aligned} &\mathrm{Card}\{X_+ \in L_+; (X_+ + Y_+, X_+ + Y_+) = A\} \\ &\quad \leq \mathrm{Card}\{X_+ \in L_+; A - (Y_+, Y_+) \leq (X_+, X_+) \leq A + (Y_+, Y_+)\} \\ &\quad \leq C \det(A)^{p+r}, \end{aligned}$$

where the last inequality follows from (i) and the fact that $\det(v)^{-r}\,\{dv\}$ is the invariant volume element on positive Hermitian matrices.

2.5. PROPOSITION. *Let $\sigma < -(p + qr)$ and $\tau = -\sigma - q + u$, $u \geq 0$. Then the series*

$$(1) \qquad \sum_{\substack{X \in L \\ (X,X) > 0}} \det(X_+, X_+)^{\sigma} \det(X, X)^{\tau} e^{-2\pi\, \mathrm{tr}(X,X)}$$

converges.

PROOF. Let $\eta = (X, X) = (X_+, X_+) - (X_-, X_-)$ and let $H_r^+(\mathcal{O})$ denote the set of positive Hermitian $r \times r$ matrices with coefficients in $\mathcal{O}$. Series (1) is majorized by

$$(2) \qquad \sum_{\substack{\eta \in H_r^+(\mathcal{O}) \\ X_- \in L_-}} \det(\eta)^{\tau} e^{-2\pi\, \mathrm{tr}(\eta)} \sum_{\substack{X_+ \in L_+ \\ (X_+, X_+) = \eta + (X_-, X_-)}} \det(X_+, X_+)^{\sigma}.$$

For a fixed η we have by Lemma 2.4(i) and 2.3

$$\begin{aligned} \sum_{X_- \in L_-} \sum_{\substack{X_+ \in L_+ \\ (X_+, X_+) = \eta + (X_-, X_-)}} \det(X_+, X_+)^{\sigma} &< \sum_{X_- \in L_-} \det(\eta + (X_-, X_-))^{\sigma+p} \\ &\leq \left(\frac{\det \eta}{\mathrm{tr}\, \eta}\right)^{\sigma+p} \sum_{X_- \in L_-} \mathrm{tr}(\eta + (X_-, X_-))^{\sigma+p} \\ &\leq (\mathrm{const.}) (\det \eta)^{\sigma+p}\, \mathrm{tr}(\eta)^{qr}. \end{aligned}$$

Series (2) is now majorized by

$$\sum_{\eta \in H_r^+(\mathcal{O})} (\mathrm{tr}\, \eta)^{qr} (\det \eta)^{\sigma+p+\tau} e^{-2\pi\, \mathrm{tr}\, \eta}.$$

Since $\sigma+p+\tau=p-q+u\geq 0$, $(\operatorname{tr}\eta)^{qr}\det(\eta)^{\sigma+p+\tau}$ is a polynomial in the entries of η and the convergence follows.

An identical argument using Lemma 2.4(ii) instead of 2.4(i) shows that for $\sigma < -(p+(q+1)r)$ and $\tau=-\sigma-q+u$, $u\geq 0$, the series

(3)
$$\sum_{\substack{X\in L\\ (X+Y,X+Y)>0}} \det((X+Y)_+,(X+Y)_+)^\sigma \det((X+Y),(X+Y))^\tau e^{-2\pi\operatorname{tr}(X+Y,X+Y)}$$

converges.

COROLLARY. *The series*

$$\sum_{X\in L}\varphi_{a,b}(X+Y)e^{-2\pi\operatorname{tr}(X+Y,X+Y)}$$

converges for $a+b>2(q+2)r$.

PROOF. Since by 1.4(1)

$$|\varphi_{a,b}(X)|\leq \det(X_+,X_+)^{-((a+b)/2+p-r)}\det(X,X)^{a+b+p-q-r},$$

we can apply (3) with $\sigma=-((a+b)/2+p-r)$ and $\tau=a+b+p-q-r$. The condition $a+b>2(q+2)r$ guarantees that $\sigma<-(p+(q+1)r)$ and $\tau=-\sigma-q+(a+b)/2$.

2.6. We consider here the convergence of the series

$$\sum_{X\in L}\varphi_{a,b}(g(X+Y))e^{-2\pi\operatorname{tr}(X+Y,X+Y)}.$$

LEMMA. *Let* $(X,X)>0$. *Then*

$$\det((gX)_+,(gX)_+)\leq C\operatorname{tr}({}^t\bar{g}g)^r\det(X_+,X_+),$$

where C is a constant independent of X or g.

PROOF. Let

$$g=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\quad X=\begin{pmatrix}X_+\\ X_-\end{pmatrix}.$$

Then $(gX)_+=AX_++BX_-$. Let $Y_+=AX_+$, $Y_-=BX_-$. Then

$${}^t\overline{(gX)}_+(gX)_+={}^t\overline{Y}_+Y_++{}^t\overline{Y}_+Y_-+{}^t\overline{Y}_-Y_++{}^t\overline{Y}_-Y_-.$$

Since ${}^t(\overline{Y_+-Y_-})(Y_+-Y_-)\geq 0$, we have

(1)
$${}^t\overline{(gX)}_+(gX)_+\leq 2({}^t\overline{Y}_+Y_++{}^t\overline{Y}_-Y_-).$$

${}^t\overline{Y}_+Y_+$, ${}^t\overline{Y}_-Y_-$ are Hermitian matrices and ${}^t\overline{Y}_+Y_+$ is positive. By simultaneous diagonalizing we conclude easily that

$$\det({}^t\overline{Y}_+Y_++{}^t\overline{Y}_-Y_-)\leq 2^r\max\{\det{}^t\overline{Y}_+Y_+,\det{}^t\overline{Y}_-Y_-\}.$$

Now

$$\det({}^t\overline{Y}_+Y_+)\leq\operatorname{tr}({}^t\bar{g}g)^r\det({}^t\overline{X}_+X_+)$$

and

$$\det({}^t\overline{Y}_-Y_-)\leq\operatorname{tr}({}^t\bar{g}g)^r\det({}^t\overline{X}_-X_-)\leq\operatorname{tr}({}^t\bar{g}g)^r\det({}^t\overline{X}_+X_+).$$

The lemma follows from 2.6(1) and the above inequalities.

COROLLARY. *The series*

$$\sum_{X\in L} \varphi_{a,b}(g(X+Y))e^{-2\pi\,\mathrm{tr}(X+Y,X+Y)}$$

converges for $a+b>2(q+2)r$.

PROOF. This follows from Corollary 2.5 and the inequality

$$\det((gX)_+,(gX)_+)^{-1}\le C\,\mathrm{tr}({}^t\bar{g}^{-1}g^{-1})^{-r}\det(X_+,X_+)^{-1}$$

which is a consequence of the lemma.

3. Some invariant theorems.

3.1. We parametrize the finite dimensional irreducible representations of $GL(p,\mathbf{C})$ by their highest weights with respect to the lower triangular Borel subgroup: $(m_1,\dots,m_p)$, $m_1\ge m_2\ge\cdots\ge m_p$, $m_i\in\mathbf{Z}$, and we denote the corresponding representation by $\lambda=\lambda(m_1,\dots,m_p)$. Let $GL(p-1,\mathbf{C})$ be embedded in $GL(p,\mathbf{C})$ as the subgroup $H_{p-1}=\{\left(\begin{smallmatrix} g & \\ & 1\end{smallmatrix}\right)\mid g\in GL(p-1,\mathbf{C})\}$.

THEOREM (3.1) [**Z**, p. 186].

$$\lambda(m_1,\dots,m_p)|_{H_{p-1}}=\sum\lambda(k_1,\dots,k_{p-1}),$$

where the summation runs over $m_1\ge k_1\ge m_2\ge k_2\ge\cdots\ge k_{p-1}\ge m_p$.

Let $\lambda_{a,b}$ be the irreducible representation of $GL(p,\mathbf{C})$ with highest weight

$$(\underbrace{a,\dots,a}_{r},0,\dots,0,\underbrace{-b,\dots,-b}_{r}),\qquad a>0,\ b>0.$$

COROLLARY. *Let* $m<p$, *and let* $GL(m,\mathbf{C})$ *be embedded in* $GL(p,\mathbf{C})$ *as the subgroup*

$$H_m=\left\{\begin{pmatrix} g & \\ & E_{p-m}\end{pmatrix}\middle| g\in GL(m,\mathbf{C})\right\}.$$

Then the multiplicity of the trivial representation occurring in $\lambda_{a,b}|_{H_m}$ *is equal to one when* $m=p-r$ *and equal to zero when* $m>p-r$.

PROOF. For an irreducible representation $\lambda(m_1,\dots,m_p)$ let $h(\lambda)$ be the number of nonzero entries among m_i. By the preceding theorem $\lambda_{a,b}|_{H_{p-1}}$ has exactly one irreducible summand τ with $h(\tau)\le 2r-2$, τ has weight

$$(\underbrace{a,\dots,a}_{r-1},0,\dots,0,\underbrace{-b,\dots,-b}_{r-1}).$$

By repeating this argument we get the conclusions of the corollary.

3.2. LEMMA [**Z**, p. 161]. *Let* λ *be an irreducible representation of* $GL(p,\mathbf{C})$ *with highest weight* $(m_1,\dots,m_p)$. *Let* $C(p)$ *be the Casimir element of* $gl(p,\mathbf{C})$ *and* $\lambda(C(p))$ *the scalar by which* $C(p)$ *acts on the space of* λ. *Then*

$$\lambda(C(p))=\sum_{i=1}^{p}m_i^2+\sum_{1\le i<j\le p}(m_i-m_j).$$

3.3. LEMMA. *Let $f(X)$ be a continuous function on $M_{p,r}(\mathbf{C})$ $(r \leq p)$ such that for $g \in U(p)$ and $\alpha \in GL(r, \mathbf{C})$*

$$f(gX\alpha) = |\det(\alpha)|^{2s} f(X).$$

Then there exists a constant C such that

$$f(X) = C \det({}^t\overline{X}X)^s.$$

PROOF. By the Gram-Schmidt process, elements of the form gdn, with $g \in U(p)$,

$$d = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_r \\ & 0 & \end{pmatrix},$$

and n upper triangular and unipotent, form a dense subset of $M_{pr}(\mathbf{C})$. Then

$$\begin{aligned} f(gdn) = f(d) &= |\det(d)|^{2s} f\left(\begin{pmatrix} E_r \\ 0 \end{pmatrix}\right) \\ &= C \det(({}^t\overline{gdn})(gdn)) \quad \text{where } C = f\left(\begin{pmatrix} E_r \\ 0 \end{pmatrix}\right). \end{aligned}$$

4. Construction of vector valued harmonic forms.

4.1. Let $\mathcal{D}$ be the symmetric space associated to the group $G = U(p,q)$. A standard realization is $\mathcal{D} = \{Z \in M_{pq}(\mathbf{C}) | (E_q - {}^t\overline{Z}Z) < 0\}$. Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$. Corresponding to the left action of G on G/K, G acts by fractional linear transformation on $\mathcal{D}$ by $g \cdot Z = (AZ+B)(CZ+D)^{-1}$. $\mathcal{D}$ can also be viewed as the space of all maximal negative subspaces of $\mathbf{C}^n$. The point Z is identified with the subspace spanned by the columns of $\left(\begin{smallmatrix} Z \\ E_q \end{smallmatrix}\right)$. The matrix $\left(\begin{smallmatrix} Z \\ E_q \end{smallmatrix}\right)$ will be denoted by $\tilde{Z}$ and we let

$$Z^\perp = \{Y \in V | (Y, \tilde{Z}) = 0\},$$

where (,) is the inner product introduced in §1.1. Given $X \in V$, its component in $Z^\perp$ is

$$X_{Z^\perp} = X - \tilde{Z}(\tilde{Z}, \tilde{Z})^{-1}(\tilde{Z}, X). \tag{1}$$

Given $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$, let

$$j(g, Z) = CZ + D.$$

Then

$$g\tilde{Z} = (\widetilde{gZ}) j(g, Z).$$

From this it follows readily that

$$(gX)_{Z^\perp} = g\left(X_{(g^{-1}Z)^\perp}\right). \tag{2}$$

4.2. Let $\bigwedge^{*,*}(\mathcal{D})_Z$ be the space of exterior differentials of bidegree $(*,*)$ at Z. The isotropy subgroup of G at Z denoted by K_Z is isomorphic to $K_{Z_0} = K = U(p) \times U(q)$ where Z_0 is the origin of $\mathcal{D}$. By choosing an orthonormal basis of cotangent space of $\mathcal{D}$ at Z, $\bigwedge^{*,*}(\mathcal{D})_Z$ is isomorphic to $\bigwedge^{*,*}(M_{pq}(\mathbf{C}))$ and furthermore the action of K_Z on $\bigwedge^{*,*}(\mathcal{D})_Z$ identifies with the representation of $U(p) \times U(q)$ on $\bigwedge^{*,*}(M_{pq}(\mathbf{C}))$. $\mathfrak{g}, \mathfrak{k}$ as before are the Lie algebras of G and K and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$

be the Cartan decomposition. Then $M_{pq}(\mathbf{C})$ is also identified with $\mathfrak{m}^*$ and the $U(p) \times U(q)$ action correspond to the dual of adjoint representation of K on $\mathfrak{m}$.

4.3. We consider a function of $X \in V$, $(X, X) > 0$, with values in an $r \times r$ matrix of differential forms of degree $(1,1)$ on $\mathcal{D}$.

$$(1) \qquad S(X, Z) = (X_{Z^\perp}, d\tilde{Z})(\tilde{Z}, \tilde{Z})^{-1}(d\tilde{Z}, X_{Z^\perp}).$$

LEMMA. *$S(gX, Z) = S(X, g^{-1}Z)$ for $g \in G$.*

PROOF. We have

$$gd(\tilde{Z}) = d(\widetilde{gZ})j(g, Z) + (\widetilde{gZ})\, dj(g, z).$$

Since $\widetilde{gZ}_{(gZ)^\perp} = 0$, we have by 4.1(2)

$$\left(X_{(g^{-1}Z)^\perp}, d(\widetilde{g^{-1}Z})\right) j(g^{-1}, Z) = \left(X_{(g^{-1}Z)^\perp}, g^{-1}d\tilde{Z}\right)$$
$$= \left(g\left(X_{(g^{-1}Z)^\perp}\right), d\tilde{Z}\right) = ((gX)_{Z^\perp}, d\tilde{Z}).$$

The lemma now follows from definition of $S(X, Z)$.

We define a differential form of degree (qr, qr) on $\mathcal{D}$ depending on $X \in V$.

$$(2) \qquad f(X, Z) = (\det S(X, Z))^q.$$

For $A \in GL(r, \mathbf{C})$ we have

$$(3) \qquad f(XA, Z) = |\det A|^{2q} f(X, Z).$$

Let

$$X_0 = \begin{pmatrix} 0 \\ E_r \\ 0 \end{pmatrix} \begin{matrix} \}p-r \\ \\ \}q \end{matrix}$$

and Z_0 as before the origin of $\mathcal{D}$. Then

$$S(X_0, Z_0) = \begin{pmatrix} dz_{p-r+1\ 1} & \cdots & dz_{p-r+1\ q} \\ \vdots & & \vdots \\ dz_{p\ 1} & \cdots & dz_{p\ q} \end{pmatrix} \begin{pmatrix} d\bar{z}_{p-r+1\ 1} & \cdots & d\bar{z}_{p1} \\ \vdots & & \vdots \\ d\bar{z}_{p-r+1\ q} & \cdots & d\bar{z}_{pq} \end{pmatrix}$$

and a simple calculation shows

$$(4) \qquad f(X_0, Z_0) = q!r! \prod_{\substack{p-r+1 \le i \le p \\ 1 \le j \le q}} dz_{ij} \wedge d\bar{z}_{ij}$$

where the products are exterior products. Let g^* be the pullback of differential forms by the map g acting on $\mathcal{D}$. By the lemma we have

$$(5) \qquad f(gX, Z) = (g^{-1})^* f(X, Z).$$

In the following, we often work with the pullback of $f(X, Z)$ to G via the projection $P\colon G \to G/K$. This pullback $\tilde{f}(X, h)$ is related to f by

$$\tilde{f}(X, h) = P^* f(X, hZ_0), \qquad h \in G.$$

We also denote by $\tilde{f}(X,h)$ the vector valued function obtained by expanding $P^*f(X,hZ_0)$ in a basis of left invariant forms in $\bigwedge^{*,*}(\mathfrak{m}^*_{\mathbf{C}})$. We then have

$$\text{(6)} \qquad \text{(i)} \quad \tilde{f}(X,hk) = \mathrm{Ad}^*(k^{-1})\tilde{f}(X,h).$$
$$\text{(ii)} \quad \tilde{f}(gX,h) = \tilde{f}(X,g^{-1}h).$$

4.4. Let $\{e_i\}$ be pluriharmonic polynomials on $M_{nr}(\mathbf{C})$ which form a basis of the representation space V_λ,

$$\lambda = (\underbrace{\alpha\cdots\alpha}_{r},\underbrace{0\cdots 0}_{n-2r},\underbrace{-\beta\cdots-\beta}_{r}), \qquad \alpha,\beta\in\mathbf{Z},\ \alpha>0,\ \beta>0.$$

Let $\{e_i^*\}$ be the dual basis for V_{λ^*} the contragredient representation. We define a function of $X \in V$, $(X,X) > 0$, with values in V_λ valued functions on $\mathcal{D}$:

$$v(X,Z) = \sum_i e_i^*(X_{Z^\perp}) \otimes e_i,$$

where $e_i^*(X_{Z^\perp})$ is the value of the pluriharmonic polynomial e_i^* on the vector $X_{Z^\perp}$. By 4.1(2) we have

$$\text{(1)} \qquad \begin{aligned} v(gX,Z) &= \sum_i e_i^*\left(g\left(X_{(g^{-1}Z)^\perp}\right)\right)\otimes e_i \\ &= \sum_i (\lambda^*(g^{-1})e_i^*)\left(X_{(g^{-1}Z)^\perp}\right)\otimes e_i \\ &= \sum_i e_i^*\left(X_{(g^{-1}Z)^\perp}\right)\otimes \lambda(g)e_i \\ &= \lambda(g)v(X,g^{-1}Z). \end{aligned}$$

It follows from [**K.V.**, III (6.3)] that

$$\text{(2)} \qquad v(XA,Z) = (\det A)^\alpha(\det \overline{A})^\beta v(X,Z).$$

We pull back $v(X,Z)$ to a V_λ valued function on G by

$$\text{(3)} \qquad \tilde{v}(X,h) = \lambda(h^{-1})v(X,hZ_0).$$

For $\tilde{v}$ we have the following properties:

$$\text{(4)} \qquad \begin{aligned} &\text{(i)} \quad \tilde{v}(X,hk) = \lambda(k^{-1})\tilde{v}(X,h), \qquad k\in K. \\ &\text{(ii)} \quad \tilde{v}(gX,h) = \tilde{v}(X,g^{-1}h). \end{aligned}$$

4.5. Consider $\tilde{f}(X,h)\tilde{v}(x,h)$. This is a vector valued form which by 4.3(6)(i) and 4.4(4)(i) satisfies

$$\tilde{f}(X,hk)\tilde{v}(X,hk) = (\mathrm{Ad}^*\otimes\lambda)(k^{-1})\tilde{f}(x,h)\tilde{v}(X,h), \qquad k\in K.$$

Let

$$\text{(1)} \qquad \bigwedge^{qr,qr}(\mathfrak{m}_{\mathbf{C}})^*\otimes V_\lambda = \bigoplus_i U_i$$

be the isotypic decomposition into irreducibles under the action of K, and let

$$\pi_0\colon \bigwedge^{qr,qr}(\mathfrak{m}_{\mathbf{C}})^*\otimes V_\lambda \to U_0$$

be the projection to the subspace U_0 with K weight

$$(\underbrace{a\cdots a}_{r},0\cdots 0,\underbrace{-b\cdots -b}_{r};\underbrace{0\cdots 0}_{q})$$

where $a=\alpha+q$ and $b=\beta+q$. Clearly U_0 has multiplicity one in the decomposition 4.5(1).

We define the vector valued form

$$\tilde{F}(X,h)=\pi_0(\tilde{f}(X,h)\tilde{v}(X,h)). \tag{2}$$

By the discussion in 4.2 there is a projection

$$(\pi_Z)_0\colon \bigwedge^{qr,qr}(\mathcal{D}_Z)\otimes V_\lambda\to (U_Z)_0$$

which corresponds to π_0 by the identification of K_Z and K, and it is straightforward to check that $\tilde{F}(X,h)$ is the pullback of the vector valued form on $\mathcal{D}$:

$$F(X,Z)=(\pi_Z)_0(f(X,Z)v(X,Z)).$$

We summarize the transformation properties of $F(X,h)$ which are consequences of preceding results.

$$\begin{aligned} &\text{(i)}\quad \tilde{F}(X,hk)=(\mathrm{Ad}^*\otimes\lambda)(k^{-1})\tilde{F}(X,h), &&k\in K.\\ &\text{(ii)}\quad \tilde{F}(gX,h)=\tilde{F}(X,g^{-1}h), &&g\in G.\\ &\text{(iii)}\quad \tilde{F}(XA,h)=\det A^a\det\overline{A}^b F(X,h), &&A\in GL(r,\mathbf{C}).\end{aligned} \tag{3}$$

4.6. We set up some terminology about pullback and pointwise inner products of vector valued forms. Associated to the finite dimensional representation $\lambda\colon G\to V_\lambda$ there is a vector bundle $E_\lambda=G\times_K V_\lambda\to\mathcal{D}$ (cf. [**M.M.**]). Condition 4.4(4)(i) says precisely that $\tilde{v}$ corresponds to a section of the bundle E_λ over $\mathcal{D}$, i.e. v. Given a map $g\colon\mathcal{D}\to\mathcal{D}$ we define a pullback g^* of the bundle E_λ

$$g^*\colon (E_\lambda)_{gZ}\to (E_\lambda)_Z,\qquad Z\in\mathcal{D},$$

given by $g^*=\lambda(g^{-1})$. This induces a pull back of sections of the bundle E_λ:

$$(g^*S)(Z)=\lambda(g^{-1})S(gZ).$$

In this terminology 4.4(1) can be stated as

$$v(gX,Z)=(g^{-1})^*v(X,Z).$$

The pullback g^* will be extended to vector valued forms and from 4.3(5) we have

$$F(gX,Z)=(g^{-1})^*F(X,Z). \tag{1}$$

E_λ has an admissible inner product $(\ ,\ )_\lambda$ (cf. [**M.M.**]) here assumed to be conjugate linear in the second factor. We let $\rho_\lambda\colon E_\lambda\to E_\lambda^*$ be the conjugate linear isomorphism on each fiber determined by

$$(u,v)_\lambda=u(\rho_\lambda(v)),\qquad u,v\in(E_\lambda)_Z.$$

To extend this inner product to vector valued forms, let $*$ be the star operator on forms defined by the invariant Kahler metric on $\mathcal{D}$. Let $\varphi,\psi\in C^\infty(\mathcal{D},\bigwedge^{k,l}(\mathcal{D})\otimes E_\lambda)$. Then

$$\rho_{\lambda^*}(\psi)\in C^\infty(\mathcal{D},\bigwedge^{pq-k,pq-l}(\mathcal{D})\otimes E_\lambda^*)$$

and

$$\varphi \wedge \rho_{\lambda^*}(\psi) \in C^\infty(\mathcal{D}, \bigwedge^{pq,pq}(\mathcal{D}))$$

where $\varphi \wedge \rho_{\lambda^*}(\psi)$ is defined to be the exterior product of differential forms together with the pairing of their vector coefficients. We use the notation

$$\varphi \wedge \rho_{\lambda^*}(\psi) = \varphi \wedge *_\lambda \psi.$$

Let $\tilde{\varphi}, \tilde{\psi}$ be their pullbacks to G. Then there is a pointwise inner product on $C^\infty(G, \bigwedge^{*,*}(\mathfrak{m}_{\mathbf{C}}^*) \otimes V_\lambda)$ (cf. [**M.M.**]) determined by

$$\varphi(Z) \wedge *_\lambda \psi(Z) = (\tilde{\varphi}(g), \tilde{\psi}(g))_\lambda \, dv_{\mathcal{D}}, \qquad Z = gZ_0.$$

4.7. We now show that $\tilde{F}(X, h)$ does not vanish identically.

PROPOSITION. *Let $e \in G$ be the identity element. Then $F(X_0, e) \neq 0$.*

PROOF. $V_\lambda | K$ contains the irreducible subspace with K weight

$$(\overbrace{\alpha, \dots, \alpha}^{r}, 0, \dots, 0, \overbrace{-\beta, \dots, -\beta}^{r}; \overbrace{0, \dots, 0}^{q}).$$

We may assume in §4.4, e_1^* is the highest weight vector given by the harmonic polynomial

$$e_1^*(X) = f_{\alpha,\beta}(X_+, \overline{X}_+),$$

where $f_{\alpha,\beta}$ is given by 1.2(2). Consider now the vector

$$\psi = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq q}} dz_{ij} \prod_{\substack{p-r+1 \leq i \leq p \\ 1 \leq j \leq q}} d\bar{z}_{ij} \otimes e_1.$$

This is a highest weight vector of weight

$$(\overbrace{\alpha + q, \dots, \alpha + q}^{r}, 0 \cdots 0, \overbrace{-\beta - q, \dots, -\beta - q}^{r}; 0, \dots, 0).$$

Let $\langle \ , \ \rangle$ be a K-invariant inner product on $\bigwedge^{qr,qr}(\mathfrak{m}_{\mathbf{C}}^*) \otimes V_\lambda$. To show that $\pi_0(\tilde{f}(X_0, e)\tilde{v}(X_0, e)) \neq 0$, it suffices to have an element $A \in U(p) \times U(q)$ such that

(1) $$\langle \tilde{f}(X_0, e)\tilde{v}(X_0, e), (\mathrm{Ad}^* \otimes \lambda)(A)(\psi) \rangle \neq 0$$

or equivalently by 4.4(4)

(2) $$\langle \mathrm{Ad}^*(A^{-1})\tilde{f}(X_0, e)\tilde{v}(A^{-1}X_0, e), \psi \rangle \neq 0.$$

We let

$$A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} E_r & & J_r \\ & \sqrt{2}E_{p-2r} & \\ -J_r & & E_r \end{pmatrix} \times E_q,$$

where J_r is the $r \times r$ matrix

$$\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

By 4.3(4)

$$\mathrm{Ad}^*(A^{-1})\tilde{f}(X_0, e) = C \prod_{\substack{1\le i\le r\\ 1\le j\le q}} dz_{ij} \wedge \prod_{\substack{p-r+1\le i\le p\\ 1\le j\le q}} d\bar{z}_{ij}$$
$$+ \text{ other terms}$$

where C is a nonzero constant. And

$$\tilde{v}((A^{-1}X_0, e)) = e_1^*((A^{-1}X_0)_+)e_1 + \cdots$$

so that the coefficient of e_1 is nonzero. Since the $\{e_i\}$ are linearly independent the desired nonvanishing (2) follows.

4.8. We can now define the basic E_λ valued differential form. Let $X \in V$, $(X, X) > 0$,

$$\omega_X^\lambda(Z) = \det(X, X)^t \det(X_{Z^\perp}, X_{Z^\perp})^s F(X, Z) \tag{1}$$

where $s = -(a + b + p - r)$ and $t = a + b + p - r - q$ as in 1.4. A consequence of 4.1(2) is

$$\det\left((gX)_{Z^\perp}, (gX)_{Z^\perp}\right) = \det\left(X_{(g^{-1}Z)^\perp}, X_{(g^{-1}Z)^\perp}\right). \tag{2}$$

This together with 4.6(1) then implies

$$\omega_{gX}^\lambda(Z) = (g^{-1})^* \omega_X^\lambda(Z). \tag{3}$$

The pullback of $\omega_X^\lambda(Z)$ to G is again denoted by $\tilde{\omega}_X^\lambda(g)$. In the following a harmonic form refers to a form which is closed and coclosed.

THEOREM. *$\omega_X^\lambda(Z)$ is a E_λ valued harmonic form of degree (rq, rq) on $\mathcal{D}$ which satisfies*

(i) *invariance under G_X;*

(ii) *has K_Z type*

$$(\underbrace{a, \ldots, a}_{r}, 0 \cdots 0, \underbrace{-b, \ldots, -b}_{r}; \underbrace{0, \ldots, 0}_{q})$$

at each $Z \in \mathcal{D}$;

(iii) *$\tilde{\omega}_X^\lambda(g)$ is square integrable over $G_X \setminus G$.*

Conversely any E valued harmonic form on $\mathcal{D}$ satisfying (i) *and* (ii) *is a constant multiple of $\omega_X^\lambda(Z)$.*

PROOF. The G_X invariance of ω_X^λ follows from (3). The K_Z type of $\omega_X^\lambda(Z)$ corresponds to the K type of $\tilde{\omega}_X^\lambda(g)$ given by 4.5(2). To prove harmonicity we first compute formally the Casimir action on $\tilde{\omega}_X^\lambda(g)$. Let $\tilde{l}(C(p,q))$ be the induced action of Casimir on $C^\infty(G, \bigwedge^{qr,qr}(\mathfrak{m}_{\mathbf{C}}^*) \otimes V_\lambda)$. By (3)

$$\tilde{l}(C(p,q))\tilde{\omega}_X^\lambda(g) = l(C(p,q))\tilde{\omega}_X^\lambda(g)$$

where $l(C(p,q))$ as in 1.1 is the action on $\tilde{\omega}_X^\lambda$ as a function of X. Since $l(C(p,q))$ commutes with left G action we need only compute at $g = e$. But by definition $\tilde{\omega}_X^\lambda(e)$ as a function of X is in the space considered in Corollary 1.3 and so

$$\tilde{l}(C(p,q))\tilde{\omega}_X^\lambda(g) = r\{(p - r + a)(a - q) + (p - r + b)(b - q)\}\tilde{\omega}_X^\lambda(g).$$

By Lemma 3.2 we also have

$$\lambda(C(p,q)) = r\{(p-r+a)(a-q)+(p-r+b)(b-q)\}.$$

Thus Kuga's lemma [**B.W.**, II] formally implies that $\tilde{\omega}_X^\lambda(g)$ is V_λ valued harmonic form. However to apply Kuga's lemma we still need to show that the coefficients of $\tilde{\omega}_X^\lambda(g)$ are in a unitary $(\mathfrak{g},\mathfrak{k})$ module. This will follow when we prove (iii).

Consider

$$\det(X,X)^{-2t}\omega_X^\lambda(Z)\wedge *_\lambda \omega_X^\lambda(Z) = \det(X_{Z^\perp},X_{Z^\perp})^{2s}F(X,Z)\wedge *_\lambda F(X,Z).$$

Let

$$F(X,Z)\wedge *_\lambda F(X,Z) = h(X,Z)\, dv_{\mathcal{D}}.$$

Since the dependence of $F(X,Z)$ on X is only on the component $X_{Z^\perp}$, $h(X,Z)$ is a function of Z and $X_{Z^\perp}$. Furthermore by 4.5(3)(iii)

$$h(XA,Z) = \det(A)^{2(a+b)}h(X,Z)$$

and clearly

$$h(kX,Z) = h(X,Z), \qquad k\in K_Z.$$

Now Lemma 3.3 implies that

$$h(X,Z) = a(Z)\det(X_{Z^\perp},X_{Z^\perp})^{a+b}.$$

By 4.5(3)(ii) $a(Z)$ must be G-invariant and thus constant. We have shown

(4) $$\det(X,X)^{-2t}\omega_X^\lambda(Z)\wedge *_\lambda \omega_X^\lambda(Z) = C\det(X_{Z^\perp},X_{Z^\perp})^{-2(p-r)-(a+b)}\, dv_{\mathcal{D}}.$$

For a given $X\in V$, $(X,X)>0$, we can find $g\in G$ and $A\in GL(r,\mathbf{C})$ such that

(5) $$g^{-1}X = X_0A.$$

Then by (2)

$$\det(X_{Z^\perp},X_{Z^\perp}) = |\det A|^2\det\left(X_{0_{(g^{-1}Z)^\perp}},X_{0_{(g^{-1}Z)^\perp}}\right).$$

Let $\Gamma_X\subset G_X$ be a uniform discrete subgroup. Then

(6) $$\int_{\Gamma_X\backslash\mathcal{D}}\det(X_{Z^\perp},X_{Z^\perp})^{-2(p-r)-(a+b)}\, dv_{\mathcal{D}} = |\det A|^{-2(2(p-r)+(a+b))}\int_{g^{-1}\Gamma_X g\backslash\mathcal{D}}\det\left(X_{0_{Z^\perp}},X_{0_{Z^\perp}}\right)^{-2(p-r)-(a+b)}\, dv_{\mathcal{D}}.$$

$g^{-1}\Gamma_X g$ is a uniform discrete subgroup of G_{X_0}, and it follows from Appendix A.2(1) that the above integral is convergent. This shows that $(\tilde{\omega}_X^\lambda,\tilde{\omega}_X^\lambda)_\lambda$ is integrable over $G_X\setminus G$ or equivalently the coefficient functions of $\tilde{\omega}_X^\lambda(g)$ are square integrable over $G_X\setminus G$. This proves (iii) and also the harmonicity of ω_X^λ.

It remains to prove the uniqueness part of the theorem. Let $\omega(Z)$ be a differential form satisfying (i) and (ii). Now $G_X\cap K_Z\simeq U(m)\times U(q)$ where $m\geq p-r$. If $m>p-r$ by Corollary 3.1 $\omega(Z)$ vanishes at Z. If $m=p-r$ we again apply Corollary 3.1 and in either case

$$\omega(Z) = b(Z)\omega_X^\lambda(Z)$$

where $b(Z)$ is a scalar function. Since $\omega(Z)$ is harmonic

$$db(Z)\wedge\omega_X^\lambda(Z)=0,\quad db(Z)\wedge *_\lambda\omega_X^\lambda(Z)=0.$$

At a given Z we expand in a basis of V_λ:

$$\omega_X^\lambda(Z)=\sum\omega_i(Z)e_i.$$

If $\omega_i(Z)\neq 0$, we can choose coordinates at Z so that $\omega_i(Z)$ and $*\omega_i(Z)$ are multiples of the complementary differentials of degree (qr,qr) and $(q(p-r),q(p-r))$. It follows immediately that $db(Z)=0$, so $b(Z)$ must be constant and the proof is finished.

5. Geometric duals to vector valued harmonic forms on $\Gamma\setminus\mathcal{D}$.

5.1. Let $k, W_k, W, V_k, (\ ,\ ), V, L_0, L, L^*$ be as in 2.2. Let G be the unitary group of $(W,(\ ,\ ))$ and

$$\Gamma=\{\gamma\in G|\gamma L_0=L_0,\ \gamma \text{ acts trivally on } L^*/L\}.$$

$\Gamma\setminus G$ is noncompact but has a finite volume. In this section to avoid complications we assume that Γ is neat which can be achieved by passing to a subgroup of finite index. The geometric interpretations can be carried over to V manifolds with additional complications. Now $\Gamma\setminus\mathcal{D}$ is smooth. Let $X\in V_k$, $(X,X)>0$, $\Gamma_X=\Gamma\cap G_X$, $\Gamma_{\langle X\rangle}=\Gamma\cap G_{\langle X\rangle}$ where $G_{\langle X\rangle}$ is the subgroup of G leaving the subspace $\langle X\rangle$ invariant. By our hypothesis on Γ we have $\Gamma_X=\Gamma_{\langle X\rangle}$. Recall that $\mathcal{D}$ may be parametrized by maximal negative subspaces in V, then we set

$$\mathcal{D}_{\langle X\rangle}=\{\langle Z\rangle \text{ maximal negative subspace } |\langle Z\rangle\perp\langle X\rangle\}.$$

then $\Gamma_{\langle X\rangle}\setminus\mathcal{D}_{\langle X\rangle}$ is also smooth and has finite volume. By 4.8(5) we have $g^{-1}X=X_0A$; it follows that $\mathcal{D}_{\langle X\rangle}=g\mathcal{D}_{\langle X_0\rangle}$. Let $P\colon\Gamma_{\langle X\rangle}\setminus\mathcal{D}\to\Gamma\setminus\mathcal{D}$ be the projection. Then $P|_{\Gamma_{\langle X\rangle}}\setminus\mathcal{D}_{\langle X\rangle}$ is generically one to one and $C_{\langle X\rangle}=P(\Gamma_{\langle X\rangle}\setminus\mathcal{D}_{\langle X\rangle})$ is an algebraic subvariety in $\Gamma\setminus\mathcal{D}$ [**T.W.4**, §4]. $\dim_{\mathbf{C}} C_{\langle X\rangle}=(p-r)q$.

5.2. Let $X\in V_k$, $(X,X)>0$ as above. We consider the series

$$\hat\omega_X^\lambda=\sum_{\gamma\in\Gamma_X\setminus\Gamma}\gamma^*\omega_X^\lambda. \tag{1}$$

LEMMA. *If $a+b>2((n+1)r-p)$ then (1) converges absolutely on $\Gamma\setminus\mathcal{D}$. If in addition $p>2r$ then (1) is square integrable on $\Gamma\setminus\mathcal{D}$.*

PROOF. By 4.8(4)

$$\omega_X^\lambda(Z)\wedge *_\lambda\omega_X^\lambda(Z)=C\det(X,X)^{-2(p+a+b-q-r)}\det(X_{Z^\perp},X_{Z^\perp})^{-2(p-r)-(a+b)}\,dv_{\mathcal{D}}.$$

Thus the square integrability of (1) will follow from that of

$$\sum_{\gamma\in\Gamma_X\setminus\Gamma}\gamma^*\det(X_{Z^\perp},X_{Z^\perp})^{-(p-r)-(a+b)/2}. \tag{2}$$

By 4.1(1) we can write

$$(X_{(gZ_0)^\perp},X_{(gZ_0)^\perp})=(X,X)+{}^t\overline{(g^{-1}x)}R_0(g^{-1}X)$$

where R_0 is a positive Hermitian matrix. We have by Lemma 2.3(i)

$$\begin{aligned}&\det((X,X)+{}^t\overline{(g^{-1}X)}R_0(g^{-1}X))\\&\quad\geq\frac{\det(X,X)}{\operatorname{tr}(X,X)}(\operatorname{tr}(X,X)+\operatorname{tr}{}^t\overline{(g^{-1}X)}R_0(g^{-1}X)).\end{aligned}$$

We can now apply [**T.W.5**, Proposition 1.14] to finish the proof.

REMARK. It follows from Corollary 2.6 that $\hat{\omega}_X^\lambda$ converges absolutely for $a+b > 2(q+2)r$ which is a better result than the above lemma when $r > 1$. We need the lemma mainly for its conclusion on square integrability.

5.3. We assume from here on in §5 that $a+b > 2((n+1)r - p)$ and $p > 2r$. Then $\hat{\omega}_X^\lambda$ is a square integrable harmonic form on $\Gamma \setminus \mathcal{D}$ with coefficients in E_λ. Its cohomology class is denoted by $[\hat{\omega}_X^\lambda] \in H^{qr,qr}(\Gamma \setminus \mathcal{D}, E_\lambda)$ where $H^{*,*}(M,E)$ refers to Dobeault cohomology with coefficients in E_λ. We discuss briefly what is meant by the geometric dual of a vector valued harmonic form. When $E_\lambda = \mathbf{C}$, the geometric dual is just the Poincaré dual. But in this case $\hat{\omega}_X$ no longer converges. For $q=1$ a method of analytic continuation was used in [**T.W.1**] to obtain harmonic forms whose Poincaré dual is the primitive component of the cycle $C_{\langle X\rangle}$. In general when $E_\lambda \neq \mathbf{C}$, and when convergence does hold the geometric dual we seek will be given by a section σ_X^λ of the bundle E_λ which is supported on $C_{\langle X\rangle}$. If $\Gamma \setminus \mathcal{D}$ is compact σ_X^λ defines a linear functional on $H^{q(p-r),q(p-r)}(\Gamma \setminus \mathcal{D}, E_\lambda^*)$ by

$$\varphi \mapsto \int_{C_{\langle X\rangle}} \sigma_X^\lambda \left(\varphi|_{C_{\langle X\rangle}}\right).$$

By Serre duality σ_X^λ determines a cohomology class in $H^{qr,qr}(\Gamma \setminus \mathcal{D}, E_\lambda)$ and we will have $[\sigma_X^\lambda] = [\hat{\omega}_X]$ which follows from the identity

$$\int_{\Gamma\setminus\mathcal{D}} \hat{\omega}_X^\lambda \wedge \varphi = \int_{C_{\langle X\rangle}} \sigma_X^\lambda \left(\varphi|_{C_{\langle X\rangle}}\right). \tag{1}$$

In the present noncompact case (1) will only be proved for φ which are square integrable and a cusp form.

5.4. Let $\bigwedge^{qr,qr}(\mathfrak{m}_{\mathbf{C}}^*) \otimes V_\lambda = \bigoplus_i U_i$ be the isotypic decomposition into irreducibles under the action of K as in 4.5(1) and let $\pi_i\colon \bigwedge^{qr,qr}(\mathfrak{m}_{\mathbf{C}}^*) \otimes V_\lambda \to U_i$ be the projections. Then the π_i commute with $(\mathrm{Ad}^* \otimes \lambda)(k)$, $k \in K$, and hence by [**M.M.**, Proposition 9.1] there is a corresponding decomposition of harmonic forms

$$H^{qr,qr}(\Gamma \setminus \mathcal{D}, E_\lambda) = \bigoplus_i H_i^{qr,qr}(\Gamma \setminus \mathcal{D}, E_\lambda)$$

where $H_i^{qr,qr}(\Gamma \setminus \mathcal{D}, E_\lambda)$ is the space of harmonic forms which has K_Z type U_i for each $Z \in \mathcal{D}$. For $\lambda = \lambda_{\alpha,\beta}$, U_0 is K type defined in §4.

Recall that the vector function $v(X,Z)$ satisfies

$$v(gX, Z) = \lambda(g) v(X, g^{-1}Z).$$

In particular for $g \in G_X$

$$v(X, Z) = \lambda(g) v(X, g^{-1}Z).$$

This shows that $v(X,Z)$ is a well-defined section of E_λ over $\Gamma_{\langle X\rangle} \setminus \mathcal{D}_{\langle X\rangle}$. Via the projection map $P\colon \Gamma_{\langle X\rangle} \setminus \mathcal{D}_{\langle X\rangle} \to C_{\langle X\rangle}$ we get a section σ_X^λ of E_λ over $C_{\langle X\rangle}$. We use the same notation σ_X^λ for the section over $\Gamma_{\langle X\rangle} \setminus \mathcal{D}_{\langle X\rangle}$. The section is locally constant. This is because of the identification of $\mathcal{D} \times V_\lambda$ with $G \times_K V_\lambda$.

5.5. LEMMA. *As in* 4.8(5) *let* $g \in G$ *and* $A \in GL(r, \mathbf{C})$ *be such that* $g^{-1}X = X_0 A$. *Then*

$$(\tilde{\omega}_X^\lambda(g), \tilde{\omega}_X^\lambda(g))_\lambda = \det(X,X)^{\alpha+\beta} C_0$$

where C_0 *is a constant independent of* X.

PROOF. By 4.8(3), 4.8(1) and 4.5(3)

$$g^*\tilde{\omega}_X^\lambda(g) = \tilde{\omega}_{g^{-1}X}^\lambda(e) = \tilde{\omega}_{X_0A}^\lambda(e) = \det A^\alpha \det \overline{A}^\beta \tilde{\omega}_{X_0}^\lambda(e).$$

Now $\det(X,X) = \det(X_0A, X_0A) = |\det(A)|^2$ and since the inner product $(\ , \ )_\lambda$ is invariant under g^*, we have

$$(\tilde{\omega}_X^\lambda(g), \tilde{\omega}_X^\lambda(g))_\lambda = \det(X,X)^{\alpha+\beta}(\tilde{\omega}_{X_0}^\lambda(e), \tilde{\omega}_{X_0}^\lambda(e))_\lambda.$$

The lemma follows with $C_0 = (\tilde{\omega}_{X_0}^\lambda(e), \tilde{\omega}_{X_0}^\lambda(e))_\lambda$.

5.6. In the following we fix the Haar measure on $G_{\langle X \rangle}$ such that

$$\operatorname{vol}(\Gamma_{\langle X \rangle} \setminus G_{\langle X \rangle}) = \operatorname{vol}(\Gamma_{\langle X \rangle} \setminus \mathcal{D}_{\langle X \rangle}).$$

LEMMA.

$$\int_{\Gamma_{\langle X_0 \rangle} \setminus G_{\langle X_0 \rangle}} (\tilde{\omega}_{X_0}^\lambda(h), \tilde{\varphi}(h))_\lambda \, dh = c \int_{\Gamma_{\langle X_0 \rangle} \setminus \mathcal{D}_{\langle X_0 \rangle}} \sigma_{X_0}^\lambda(*_\lambda \varphi)$$

where $\tilde{\varphi}(h) \in U_0$, for each $h \in G$ and c is a constant independent of φ.

PROOF. Since $\tilde{\varphi}(h) \in U_0$, we have

$$\begin{aligned}(\tilde{\omega}_{X_0}^\lambda(h), \tilde{\varphi}(h))_\lambda &= (\pi_0(\tilde{f}(X_0,h)\tilde{v}(X_0,h)), \tilde{\varphi}(h))_\lambda \\ &= (\tilde{f}(X_0,h)\tilde{v}(X_0,h), \tilde{\varphi}(h))_\lambda.\end{aligned}$$

And by definitions,

(1) $$(\tilde{f}(X_0,h)\tilde{v}(X_0,h), \tilde{\varphi}(h))_\lambda \, dv_{\mathcal{D}} = f(X_0,Z)v(X_0,Z) \wedge *_\lambda \varphi(Z)$$

with $Z = hZ_0$. Now let dv_F be the volume element along the fibers as in A.2. Then by 4.3(4)

$$f(X_0, Z_0) = c(dv_F)_{Z_0},$$

and since both are $G_{\langle X_0 \rangle}$ invariant and $G_{\langle X_0 \rangle}$ acts transitively on $\mathcal{D}_{\langle X_0 \rangle}$ we conclude

$$f(X_0, Z) = c\, dv_F \quad \text{on } \mathcal{D}_{\langle X_0 \rangle}.$$

It follows that along $\mathcal{D}_{\langle X_0 \rangle}$

(2) $$f(X_0,Z)v(X_0,Z) \wedge *_\lambda\varphi(Z) = f(X_0,Z)v(X_0,Z) \wedge (*_\lambda \left(\varphi(Z)|_{\mathcal{D}_{\langle X_0 \rangle}}\right),$$

where $*_\lambda\varphi(Z)|_{\mathcal{D}_{\langle X_0 \rangle}}$ denotes the pullback of this form to $\mathcal{D}_{\langle X_0 \rangle}$. This is because the components of $*_\lambda\varphi(Z)$ with differentials in the direction of F are annihilated in the exterior product with $f(X_0, Z)$. Next by A.2, we also have

(3) $$dv_{\mathcal{D}} = dv_F \wedge dv_{\mathcal{D}_{\langle X_0 \rangle}} \quad \text{along } \mathcal{D}_{\langle X_0 \rangle}.$$

(Note that this is an equality of the volume forms along $\mathcal{D}_{\langle X_0 \rangle}$, not the pullback of these forms to $\mathcal{D}_{\langle X_0 \rangle}$.)

From (1)–(3) it follows that

$$(\tilde{f}(X_0,h)\tilde{v}(X_0,h), \tilde{\varphi}(h))_\lambda \, dv_{\mathcal{D}_{\langle X_0 \rangle}} = cv(X_0,Z)\left(*_\lambda\varphi(Z)|_{\mathcal{D}_{\langle X_0 \rangle}}\right)$$

and the lemma follows.

5.7. We now prove the identity 5.3(1). It will suffice to assume $a + b > 2(q+2)r$ to have absolute convergence of $\hat{\omega}_X^\lambda$ (cf. Proposition 6.2 and 6.3(3)).

THEOREM. *Let φ be a harmonic cusp form, $[\varphi] \in H^{qr,qr}(\Gamma \backslash D, E_\lambda)$. Let $a+b > 2(q+2)r$. Then*

$$\int_{\Gamma\backslash D} \hat{\omega}_X^\lambda \wedge *_\lambda \varphi = C_2 \int_{C_{\langle X \rangle}} \sigma_X^\lambda \left(*_\lambda \varphi|_{C_{\langle X \rangle}}\right)$$

where C_2 is a universal nonzero constant.

PROOF. We first claim that by our hypothesis

$$\int_{\Gamma_{\langle X \rangle}\backslash D} |\omega_X^\lambda \wedge *_\lambda \varphi| < \infty.$$

This is because $\|\varphi\|$ is bounded while $\|\omega_X^\lambda\| \le \det(X_{Z^\perp}, X_{Z^\perp})^{-(p-r)-(a+b)/2}$ by 4.8(4). By Appendix A.2(2) the above integral converges if $-(p-r) - \frac{1}{2}(a+b) + p+q < 1$. This follows easily from the hypothesis on $a+b$. The unfolding argument then implies

$$\begin{aligned}(1)\qquad \int_{\Gamma\backslash D} \hat{\omega}_X^\lambda \wedge *_\lambda \varphi &= \int_{\Gamma_{\langle X \rangle}\backslash D} \omega_X^\lambda \wedge *_\lambda \varphi = \int_{\Gamma_{\langle X \rangle}\backslash G} (\tilde{\omega}_X^\lambda, \tilde{\varphi})_\lambda \, dg \\ &= \int_{G_{\langle X \rangle}\backslash G} \left(\int_{\Gamma_{\langle X \rangle}\backslash G_{\langle X \rangle}} (\tilde{\omega}_X^\lambda(hg), \tilde{\varphi}(hg))_\lambda \, dh \right) d\mu(g) \\ &= \int_{G_{\langle X \rangle}\backslash G} \left(\tilde{\omega}_X^\lambda(g), \int_{\Gamma_{\langle X \rangle}\backslash G_{\langle X \rangle}} \tilde{\varphi}(hg) \, dh \right)_\lambda d\mu(g)\end{aligned}$$

where the last equality follows from the G_X invariance of $\tilde{\omega}_X^\lambda(g)$. Note that $(\tilde{\omega}_X^\lambda, \tilde{\varphi})_\lambda$ depends only on the component of $\tilde{\varphi}$ which has values in U_0. So we may assume $\tilde{\varphi}(g) \in U_0$, $g \in G$. Let

$$\tilde{\Phi}_X(g) = \int_{\Gamma_{\langle X \rangle}\backslash G_{\langle X \rangle}} \tilde{\varphi}(hg) \, dh.$$

Then $\tilde{\Phi}_X(g)$ is a harmonic form satisfying conditions (i), (ii) of Theorem 4.8. We therefore conclude

$$\tilde{\Phi}_X(g) = C_X \tilde{\omega}_X^\lambda(g).$$

We assume temporarily that $X = X_0$. Then by Lemma 5.6

$$\int_{\Gamma_{\langle X_0 \rangle}\backslash G_{\langle X_0 \rangle}} (\tilde{\omega}_{X_0}^\lambda(h), \tilde{\varphi}(h))_\lambda \, dh = C \int_{\Gamma_{\langle X_0 \rangle}\backslash D_{\langle X_0 \rangle}} \sigma_{X_0}^\lambda \left(*_\lambda \varphi|_{D_{\langle X_0 \rangle}}\right)$$

and it follows that

$$\overline{C}_{X_0}(\tilde{\omega}_{X_0}^\lambda(e), \tilde{\omega}_{X_0}^\lambda(e)) = C \int_{\Gamma_{\langle X_0 \rangle}\backslash D_{\langle X_0 \rangle}} \sigma_{X_0}^\lambda \left(*_\lambda \varphi|_{D_{\langle X_0 \rangle}}\right).$$

This shows that up to a universal constant C_1 which is independent of φ

$$\overline{C}_{X_0} = C_1 \int_{\Gamma_{\langle X_0 \rangle}\backslash D_{\langle X_0 \rangle}} \sigma_{X_0}^\lambda \left(*_\lambda \varphi|_{D_{\langle X_0 \rangle}}\right).$$

Now (1) may be evaluated by

$$(3)\qquad \int_{\Gamma_{\langle X_0\rangle}\backslash \mathcal{D}} \omega_{X_0}^\lambda \wedge *_\lambda \varphi = \overline{C}_{X_0} \int_{G_{\langle X_0\rangle}\backslash G} (\tilde{\omega}_{X_0}^\lambda(g), \tilde{\omega}_{X_0}^\lambda(g))_\lambda \, d\mu(g)$$
$$= C_2 \int_{\Gamma_{\langle X_0\rangle}\backslash \mathcal{D}_{\langle X_0\rangle}} \sigma_{X_0}^\lambda \left(*_\lambda \varphi|_{\mathcal{D}_{\langle X_0\rangle}}\right)$$

where C_2 is another universal constant. This proves the theorem for $X = X_0$. The general case reduces to this special case by the identities:

$$g^*\omega_X^\lambda = (\det A)^\alpha (\det \overline{A})^\beta \omega_{X_0}^\lambda,$$
$$g^*\sigma_X^\lambda = (\det A)^\alpha (\det \overline{A})^\beta \sigma_{X_0}^\lambda$$

where $g^{-1}X = X_0 A$, so that

$$\int_{\Gamma_{\langle X\rangle}\backslash \mathcal{D}} \omega_X^\lambda \wedge *_\lambda \varphi = \int_{\Gamma_{\langle X_0\rangle}\backslash \mathcal{D}} (g^*\omega_X^\lambda) \wedge *_\lambda (g^*\varphi)$$
$$= \det A^\alpha \det \overline{A}^\beta \int_{\Gamma_{\langle X_0\rangle}\backslash \mathcal{D}} \omega_{X_0}^\lambda \wedge *_\lambda (g^*\varphi)$$
$$= C_2 \det A^\alpha \det \overline{A}^\beta \int_{\Gamma_{\langle X_0\rangle}\backslash \mathcal{D}_{\langle X_0\rangle}} \sigma_{X_0}^\lambda (*_\lambda g^*\varphi)$$
$$= C_2 \int_{\Gamma_{\langle X_0\rangle}\backslash \mathcal{D}_{\langle X_0\rangle}} (g^*\sigma_X^\lambda)(*_\lambda g^*\varphi)$$
$$= C_2 \int_{\Gamma_{\langle X\rangle}\backslash \mathcal{D}_{\langle X\rangle}} \sigma_X^\lambda (*_\lambda \varphi).$$

6. Theta functions and correspondence.

6.1. Recall [**T.W.5**, §3] that the symmetric space of $U(J_r)$ can be realized as

$$\mathcal{H}_r = \{\tau \in M_{rr}(\mathbf{C}) | \tau - \tau^*/\sqrt{-1} > 0\}.$$

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in U(J_r)$ where $a, b, c, d \in M_{rr}(\mathbf{C})$. Then g acts on $\mathcal{H}_r$ by

$$g\tau = (a\tau + b)(c\tau + d)^{-1}.$$

An element $\tau \in \mathcal{H}_r$ can be expressed uniquely as

$$\tau = u + \sqrt{-1}v, \quad \text{with } u = u^*,\ v = v^*,\ v > 0.$$

Let

$$\sigma_\tau = \begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix} \in U(J_r).$$

Then $\sigma_\tau \cdot (\sqrt{-1}I_r) = \tau$.

We have automorphic factors

$$J(g, \tau) = (J_1(g, \tau), J_2(g, \tau))$$

where

$$J_1(g, \tau) = a - (g\tau)c, \quad J_2(g, \tau) = c\tau + d,$$

and the identities

$$J(\kappa(x,y),\sqrt{-1}I_r) = (x,y),$$
$$J(\sigma_\tau,\sqrt{-1}I_r) = (v^{1/2}, v^{-1/2}).$$

6.2. Consider the function (cf. 1.5)

$$l(g)\varphi_{a,b}(X)e^{-2\pi\,\mathrm{tr}(X,X)} = \varphi_{a,b}(g^{-1}X)e^{-2\pi\,\mathrm{tr}(X,X)}.$$

Since $l(g)$ commutes with $r_0(g)$ we have by Corollary 1.5

$$\text{(1)}\qquad \begin{aligned} &r_0(\kappa(x,y))\varphi_{a,b}(g^{-1}X)e^{-2\pi\,\mathrm{tr}(X,X)} \\ &\quad = \det(x)^{a-q}\det(y)^{-p-b}\varphi_{a,b}(g^{-1}X)e^{-2\pi\,\mathrm{tr}(X,X)}. \end{aligned}$$

Let $\alpha(x,y) = \det(x)^{a-q}\det(y)^{-p-b}$. Then from 6.1

$$\text{(2)}\qquad \alpha(J(\sigma_\tau,\sqrt{-1}I_r)^{-1}) = (\det v)^{(q-p-a-b)/2}.$$

As in §2.2 let $\mathcal{O}$ be the ring of integers of the imaginary quadratic field k and L, L^* be the $\mathcal{O}$ lattices. Let N be a positive integer such that $NL^* \subset L$, $N(XX) \equiv 0\ (2\mathcal{O})$ for $X \in L^*$ and let

$$\Gamma'(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in U(J_r)(\mathcal{O}) | a \equiv d \equiv I_r(N),\ b \equiv c \equiv 0\ (N)\right\}.$$

We denote by $\chi(\gamma)$ the character of $\Gamma'(N)$ given by (cf. [**T.W.5**, 11.7(1)])

$$\chi(\gamma) = \varepsilon(\gamma)^{-1}|\det(d)|^{-n}\sum_{l\in L/Ld^*} e[\mathrm{tr}(l, lbd^{-1})]$$

where

$$\varepsilon(\gamma) = \begin{cases} i^{-r(p-q)}\det(c)^n|\det(c)|^{-n} & \text{if } \det(c) \neq 0, \\ |\det(a)|^n\det(a^*)^{-n} & \text{if } c = 0\end{cases}$$

and $e[x] = e^{2\pi i x}$.

PROPOSITION. *Let $a+b > 2(q+2)r$ and define*

$$\theta(\lambda,\tau,h,Z) = \sum_{X\equiv h(L)} \omega_X^\lambda(Z)e[\mathrm{tr}(X,X)\tau].$$

Then for $\gamma \in \Gamma'(N)$

$$\theta(\lambda,\gamma\tau,h,Z) = \chi(\gamma)\alpha(J(\gamma,\tau)^{-1})\theta(\lambda,\tau,h,Z)$$

where $\alpha(J(\gamma,\tau)^{-1}) = \det J_1(\gamma,\tau)^{q-a}\det J_2(\gamma,\tau)^{p+b}$.

PROOF. Let $gZ_0 = Z$ and set

$$\theta_1(\lambda,\tau,h,Z) = \sum_{X\equiv h(L)} (\det v)^{(q-p-a-b)/2} r_0(\sigma_\tau)\varphi_{a,b}(g^{-1}X)e^{-2\pi\,\mathrm{tr}(X,X)}.$$

Then we claim that for $\gamma \in \Gamma'(N)$

$$\text{(3)}\qquad \theta_1(\lambda,\gamma\tau,h,Z) = \chi(\gamma)\alpha(J(\gamma,\tau)^{-1})\theta_1(\gamma,\tau,h,Z).$$

This transformation formula follows formally from 6.2(1) and [**T.W.2**, Proposition 1.5]. The only difference is that $\varphi_{a,b}(g^{-1}X)e^{-2\pi(X,X)}$ is not a Schwartz function in X since (X,X) is indefinite. However, the necessary convergence requirements for θ_1 has been shown in §2. In particular by Corollary 2.6 $\varphi_{a,b}(g^{-1}X)e^{-2\pi\,\mathrm{tr}(X,X)}$ satisfies the Poisson summation formula [**R.S.2**, p. 1060, (A), (B)]. The transformation can then be derived as for Schwartz functions.

Note that the key step in the proof of [**R.S.2**, Theorem 1.5] is the convergence of their (1.19). But this is the content of our Corollary 2.6. We refer to [**R.S.2**] for details which are omitted here.

Next by [**T.W.2**, (1.10)]

$$\begin{aligned} &r_0(\sigma_\tau)\varphi_{a,b}(g^{-1}X)e^{-2\pi\,\mathrm{tr}(X,X)} \\ &\qquad = \det(v)^{(p-q+a+b)/2}\varphi_{a,b}(g^{-1}X)e^{2\pi i\,\mathrm{tr}(X,X)\tau}. \end{aligned}$$

Thus

$$\theta_1(\lambda,\tau,h,gZ_0) = \sum_{X\equiv h(L)} \varphi_{a,b}(g^{-1}X)e^{2\pi i\,\mathrm{tr}(X,X)\tau}.$$

Finally since $\omega_X^\lambda(gZ_0) = \omega_{g^{-1}X}^\lambda(Z_0)$, and $\omega^\lambda(Z_0)$ as a function of X is contained in the K span of $\varphi_{a,b}(X)$, the proof of the proposition is complete.

6.3. Let

$$\begin{aligned} H_r(k) &= \{r\times r \text{ Hermitian matrices with coefficients in } k\}, \\ H_r(N\mathcal{O}) &= \{r\times r \text{ Hermitian matrices with coefficients in } N\mathcal{O}\}, \\ H_r^*(N\mathcal{O}) &= \{\eta\in H_r(k) \mid \mathrm{tr}(\eta H_r(N\mathcal{O}))\subset \mathbf{Z}\}. \end{aligned}$$

Given $\eta\in H_r^*(N\mathcal{O})$, $\eta>0$ and $h\in L^*$, we have a decomposition into finitely many Γ orbits.

$$L_{\eta,h} = \{X\in L^*, X\equiv h(L) \mid (X,X)=\eta\} = \bigcup_{i=1}^{l}\Gamma X_i.$$

We denote

(1) $$C_\eta = \sum_{i=1}^{l} C_{\langle X_i\rangle}.$$

Similarly we set

(2) $$\hat\omega_\eta^\lambda = \sum_{X\in L_{\eta,h}} \omega_X^\lambda.$$

Then

$$\hat\omega_\eta^\lambda = \sum_{i=1}^{l}\hat\omega_{X_i}^\lambda$$

where

$$\hat\omega_{X_i}^\lambda = \sum_{\gamma\in\Gamma_{X_i}\backslash\Gamma}\gamma^*\omega_{X_i}^\lambda.$$

By Theorem 5.7 the cohomology class of $\hat\omega_\eta^\lambda$ is represented (in terms of currents) as a sum of the sections $\sigma_{X_i}^\lambda$ supported over the cycles $C_{\langle X_i\rangle}$, $1\le i\le l$. We can rewrite the summations in $\theta(\lambda,\tau,h,Z)$ in the form

(3) $$\theta(\lambda,\tau,h,Z) = \sum_{\substack{\eta\in H_r^*(N\mathcal{O})\\ \eta>0}} \hat\omega_\eta^\lambda(Z)e[\mathrm{tr}(\eta\tau)],$$

where the fact that we sum over $\eta \in H_r^*(N\mathcal{O})$ follows from the transformation formula of Proposition 6.2.

The correspondence of Hermitian modular forms for $\Gamma'(N)$ and harmonic forms on $\Gamma \setminus \mathcal{D}$ follows immediately from this formula. Namely let $\mathcal{S}_{\alpha,\beta}(\Gamma'(N),\chi)$ be the space of holomorphic cusp forms on $\mathcal{H}_r$ satisfying, for $\gamma \in \Gamma'(N)$,

$$f(\gamma\tau) = \chi(\gamma)\det J_1(\gamma,\tau)^{\alpha}\det J_2(\gamma,\tau)^{\beta} f(\tau).$$

Then (cf. [**T.W.2**, §3]) if $\alpha+\beta > 4r-2$ the Poincaré series associated to $\eta \in H_r^*(N\mathcal{O})$, $\eta > 0$,

$$\phi_\eta(\tau) = \sum_{\gamma \in \Gamma'_\infty \setminus \Gamma'(N)} \chi(\gamma)^{-1}\det J_1(\gamma,\tau)^{-\alpha}\det J_2(\gamma,\tau)^{-\beta} e[\operatorname{tr}\eta\gamma\tau],$$

$$\Gamma'_\infty = \Gamma'(N) \cap \left\{ \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix} \middle| \beta \in H_r(N\mathcal{O}) \right\},$$

is absolutely convergent. Furthermore $\phi_\eta(\tau)$, $\eta > 0$, span the space $\mathcal{S}_{\alpha,\beta}(\Gamma'(N),\chi)$. The Peterson inner product on $\mathcal{S}_{\alpha,\beta}(\Gamma'(N),\chi)$ is given by

$$\langle \phi_1, \phi_2 \rangle = \int_{\Gamma'(N)\setminus \mathcal{H}_r} \overline{\phi_1(\tau)}\phi_2(\tau)\det v^{\alpha-\beta-2r}\{du\}\{dv\}.$$

A form $\phi \in \mathcal{S}_{\alpha,\beta}(\Gamma'(N),\chi)$ has Fourier expansion

$$\phi(\tau) = \sum_{\substack{\eta \in H_r^*(N\mathcal{O}) \\ \eta > 0}} a(\eta) e[\operatorname{tr}(\eta\tau)]$$

and a standard property of Poincaré series (cf. [**T.W.2**, (3.13)] gives

$$\langle \phi_\eta, \phi \rangle = C \det \eta^{\alpha-\beta} a(\eta) \tag{4}$$

where C is a universal constant independent of ϕ and η. This same calculation shows that if $\alpha = a - q$, $\beta = b + p$, then

$$\langle \phi_\eta, \theta(\lambda,\tau,h,Z) \rangle = C \det \eta^{q-p-a-b} \hat{\omega}_\eta^\lambda(Z). \tag{5}$$

We summarize the preceding discussion in the following theorem.

THEOREM. *Let $a+b > 2(q+2)r$. Then there is a lifting*

$$\mathbf{L}\colon \mathcal{S}_{a-q,b+p}(\Gamma'(N),\chi) \to H^{qr,qr}(\Gamma \setminus \mathcal{D}, E_\lambda)$$

given by

$$\mathbf{L}(\phi) = \langle \phi, \theta(\lambda,\tau,h,Z) \rangle.$$

In particular on a Poincaré series ϕ_η, the image is

$$\mathbf{L}(\phi_\eta) = C \det \eta^{q-p-a-b} \hat{\omega}_\eta^\lambda$$

where C is a nonzero constant independent of η. $\hat{\omega}_\eta^\lambda$ is square integrable when $p > 2r$ and $a+b > 2((n+1)r-p)$.

REMARK. The results in [**T.W.2**, **4**] correspond to the case $a = b = q = 1$ and the result in [**T.W.3**] corresponds to the case $a = b = q = 2$, $r = 1$. The convergence of $\theta(\gamma,\tau,h,Z)$ fails in these cases and that is one of the reasons for the analytic continuation methods used in [**T.W.3**]. In [**T.W.4**] a different approach is used to directly construct the adjoint of **L** which bypasses the analytic continuations. However, none of these previous methods allows us to conclude the nonvanishing of these liftings.

6.4. Let $a+b > 2(q+2)r$. Then $\omega_X^\lambda(Z)e[\mathrm{tr}(X,X)\tau]$ is absolutely summable over the lattice L_0^r, and so by [**T.W.5**, Lemma 13.24] there exist a multiple L of L_0^r and $h \in L_0^r/L$ such that

$$\theta(\lambda,\tau,h,Z) = \sum_{X\equiv h(L)} \omega_X^\lambda(Z)e[\mathrm{tr}(X,X)\tau] \neq 0.$$

This implies by 6.3(3) that

$$\sum_{\substack{\eta\in H_r^*(N\mathcal{O})\\ \eta>0}} \hat{\omega}_\eta^\lambda(Z)e[\mathrm{tr}(\eta\tau)] \neq 0.$$

Hence there exists $\eta \in H_r^*(N\mathcal{O})$, $\eta > 0$, such that $\hat{\omega}_\eta^\lambda \neq 0$. Once L is chosen we also have L^*, and then Γ is determined as in 5.1. $\hat{\omega}_\eta^\lambda$ is thus a nonzero harmonic form on $\Gamma \setminus \mathcal{D}$. Similarly $\Gamma'(N)$ is determined as in 6.2. Theorem 6.3 then shows that $\phi_\eta \neq 0$.

PROPOSITION. *Let $a+b > 2(q+2)r$. Then there exist a multiple L of L_0^r, $h \in L_0^r/L$, and $\eta \in H_r^*(N\mathcal{O})$ such that $\hat{\omega}_\eta^\lambda$ is a nonzero harmonic form on $\Gamma \setminus \mathcal{D}$ where $\Gamma = \{\gamma \in G|\gamma L_0 = L_0, \gamma$ acts trivially on $L^*/L\}$.*

Appendix. We prove two technical results which are used in the paper.

A.1. As in §2.1 let $X \in M_{pr}(\mathbf{C}), \{dX\}$ the Euclidean volume element and $Y - {}^t\overline{X}X = (X,X) \in H_r^+$ the domain of positive Hermitian $r \times r$ matrices. $\{dY\}$ is the corresponding Euclidean volume element. Let f be a function of (X,X).

LEMMA.

$$\int_{\substack{X\in M_{pr}(\mathbf{C})\\ (X,X)>0}} f((X,X))\,\{dX\} = C_1 \int_{Y\in H_r^+} f(Y)\det Y^{p-r}\,\{dY\}.$$

PROOF. For any $X \in M_{pr}(\mathbf{C})$, there exist $\tau \in U(p)$ and $\sigma \in U(r)$ such that

$$X = \tau\begin{pmatrix} D(\lambda) \\ 0 \end{pmatrix}\sigma \quad \text{where } D(\lambda) = \mathrm{diag}(\lambda_1,\ldots,\lambda_r).$$

In terms of this decomposition we have (cf. [**W.1**, 5.11])

$$\{dX\} = C\prod_{i=1}^r \lambda_i^{2(p-r)} \prod_{1\le i<j\le r} (\lambda_i^2-\lambda_j^2)^2 \prod_{i=1}^r d(\lambda_i^2)\,d\sigma\,d\tau.$$

Now $Y = \sigma^* D(\lambda^2)\sigma$ and by [**H**, (3.3.2)]

$$\{dY\} = \prod_{1\le i<j\le r} (\lambda_i^2-\lambda_j^2) \prod_{i=1}^r d(\lambda_i)^2\,d\sigma.$$

This proves the lemma.

A.2. Notations are as in 4.8(6). We consider the convergence of

$$(1) \qquad \int_{\Gamma_{X_0}\setminus\mathcal{D}} \det\left(X_{0_{Z^\perp}}, X_{0_{Z^\perp}}\right)^{-2(p-r)-(a+b)} dv_{\mathcal{D}}$$

where Γ_{X_0} is a uniform discrete subgroup of G_{X_0}. The argument is identical to that of the real case [**W.1**, (4.16)]. First of all we have

$$\det\left(X_{0_{Z^\perp}}, X_{0_{Z^\perp}}\right) = \frac{B(Z)}{A(Z)}$$

where $A(Z) = \det(E - {}^t Z\overline{Z})$ and $B(Z) = \det(E - {}^t Z_1\overline{Z}_1)$. Here we let

$$Z_1 = \begin{pmatrix} z_{11} & \cdots & z_{1q} \\ \vdots & & \vdots \\ z_{p-r-1} & \cdots & z_{p-r\ q} \\ & 0 & \end{pmatrix}$$

and

$$Z_2 = \begin{pmatrix} & 0 & \\ z_{p-r+1\ 1} & \cdots & z_{p-r+1\ q} \\ \vdots & & \vdots \\ z_{p\ 1} & \cdots & z_{pq} \end{pmatrix}.$$

Next

$$dv_{\mathcal{D}} = (B/A)^{p-r}\, dv_{\mathcal{D}_{\langle X_0\rangle}}\, dv_F$$

where $dv_{\mathcal{D}_{\langle X_0\rangle}} = (B)^{-(p+q-r)}\,\{dZ_1\}$ is an invarinat volume element of $\mathcal{D}_{\langle X_0\rangle}$, and

$$dv_F = A^{-r}(B/A)^q\,\{dZ_2\}$$

is an invariant volume element along the fibers of $\mathcal{D} \to \mathcal{D}_{\langle X_0\rangle}$. Since (A/B) and dv_F are G_{X_0} invariant

$$(2)\qquad \int_{\Gamma_{X_0}\backslash\mathcal{D}} \left(\frac{B}{A}\right)^s dv_{\mathcal{D}} = \int_{\Gamma_{X_0}\backslash\mathcal{D}} \left(\frac{B}{A}\right)^{s+p-r} dv_{\mathcal{D}_{\langle X_0\rangle}}\, dv_F$$
$$= \operatorname{vol}(\Gamma_{X_0}\setminus\mathcal{D}_{\langle X_0\rangle}) \int_{\mathcal{D}_2} \det(E - {}^t Z_2\overline{Z}_2)^{-(s+p+q)}\,\{dZ_2\}$$

where $\mathcal{D}_2 = \{Z_2 | {}^t Z_2\overline{Z}_2 < E_q\}$. By [**H**, p. 40] the integral converges if $s+p+q<1$. For applications to (1) above we have $s = -2(p-r)-(a+b)$ which guarantees the inequality since $p \geq 2r$ and $a+b \geq 2q \geq 2$.

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